RESEARCH ARTICLE

A numerical approach for solving a class of fractional optimal control problems using Genocchi polynomials

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ABSTRACT

In this paper, an efficient and accurate computational method based on the Genocchi polynomials is proposed for solving a class of fractional optimal control problems. In the proposed method, the Caputo fractional derivative operator for the Genocchi polynomials is given. The proposed technique is applied to transform the state and control variables into non-linear programing parameters at collocation points. The most important advantages of our method are easy implementation, simple operations. Some illustrative examples are presented to show the efficiency and accuracy of the method.

Keywords: Fractional optimal; Control problems; Caputo derivative; Genocchi polynomials; Operational matrix; Non-linear programming

INTRODUCTION

In the present paper, we consider a class of optimal control problems the objective function and the dynamic system with the Caputo fractional derivative as follows:

$$J[x,u]=\int_0^1 h(x(t),u(t),t)dt, \qquad (1)$$

$$ {}_0^C D_t^\beta x(t)=f(x(t),u(t),t), \qquad (2)$$

$$g(x(t),u(t),t)\leq 0, \qquad (3)$$

$$x(0)=x_0. \qquad (4)$$

where h is a scalar function, x(t) is state vector and u(t) is control vector of dimension n × 1 and m × 1, respectively [1-3]. In the real word, many physical phenomena are controlled by the differential equations. Therefore, in recent years, Optimal Control Problems (OCPs) have been the interest of many scientists. A lot of research has been done in the context of OCPs, but the research on the Fractional Optimal Control Problems (FOCPs) is not so high. Ashpazzadeh, et al., in have used Hermite spline multiwavelest to solve the FOCPs. In, the ractional Remann-Liuovel is used to numerically solve the problem. Also, is a used numerical simulation for FOCPs with the Caputo fractional derivative in. In, Legandar functions are used as basis for solving FOCPs. Keshavarze used Bernoulli,s polynomials to solve FOCPs. Mashayekhi, et al., used hebrid functions basis to numerically solve FOCPs. The main aim of this paper is to solve fractional optimal control problems in the sense of Caputo derivative by using Genocchi polynomials. With the help of the Genocchi polynomials, the objective function, state and control vectors are expanded. To calculate coefficients, we used the collocation method with the nodes in the Chebyshev roots as collocation points. In finally, the FOCPs transformed into a problem with algebraic equations that can be solved by suitable algorithm. The paper is organized as follows: In the next section, we introduce the preliminary integration and fractional derivative. In section 3, we describe the basic formulating of the Genocchi polynomials required for our subsequent development. I section 4, we apply the Genocchi polynomials on [0, 1] to solve equations (1)-(4). In section 5, we will solve two numerical examples with the proposed method [4-6].

MATERIALS AND METHODS

Some preliminaries in fractional calculus:

Here, we give two definitions related to Rieman-Liouvill fractional integral and Caputo fractional derivative.

Definition 1: The Riemann-Liouvill fractional integral of order β>0 of a function f is de- fined as follows:

$$ {}_0I_t^\beta f(t)=\begin{cases}\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}f(\tau)d\tau, & \beta>0,\\ f(t), & \beta=0.\end{cases} \qquad (5)$$

which I^β is called the Riemann-Liouville fractional integration operator. Caputo,s derivative operator D^β of a function f (t) is defined as follow:

$$ {}_0^C D_t^\beta f(t)=\frac{1}{\Gamma(n-\beta)}\int_a^t (t-\tau)^{n-\beta-1}f^{(n)}(\tau)d\tau.$$

Definition 2: The Caputo fractional derivative of order β with the lower limit zero for a function

$f\in C^n(0,\infty)$ is defined as follows. Some properties of Caputo, s fractional derivatives as:

$$ {}_0^C D_t^\beta c=0, \quad (cisconstant)$$

$$ {}_0^C D_t^\beta t^\alpha=\begin{cases}0, \alpha\in N\cup\{0\}, \alpha<\lceil\beta\rceil\\ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\beta)}t^{\alpha-\beta}, & \alpha\in\mathbb{N}\cup\{0\}, \alpha\geq\lceil\beta\rceil \; or \; \alpha\notin\mathbb{N}, \alpha>\lfloor\beta\rfloor\end{cases}$$

Where ⌈.⌉ is the ceiling function. Also the Caputo fractional-order derivative operator is a linear operator. That is, for all real scalers λ and μ and for all functions f (t) and g(t), we have:

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$$ {}_{a}^{C}D_{t}^{\alpha}(\lambda f(t)+\mu g(t))=\lambda_{a}^{C}D_{t}^{\alpha}f(t)+\mu_{a}^{C}D_{t}^{\alpha}g(t) $$

RESULTS AND DISCUSSION

Properties of Genocchi polynomials

The genocchi polynomials: Suppose $G_i(t)$ is the Genocchi polynomials that obtained from the formula below:

$$ G_i(t)=\sum_{r=0}^{i}\binom{i}{r}G_{i-r}t^r, \tag{6} $$

where G_{i-r}, r = 0, 1, . . . , i in Eq. (6) are the Genocchi numbers, that can be found as:

$$ \begin{aligned} &G_0=0,\\ &G_1=1,\\ &G_i=2(1-2^i)B_i, \end{aligned} $$

where B_i is the Euler, s numbers, which is defined as:

$$ \begin{aligned} &B_0=1\\ &B_1=-\frac{1}{2}\\ &B_{2i+1}=0 \quad i=1,2,\dots\\ &B_i(0)=B_i(1) \quad i\neq 1, \end{aligned} $$

And

$$ B_i(t)=\sum_{r=0}^{i}\binom{i}{r}B_i t^{i-r}, $$

Thus, using Eq (6) and Genocchi numbers, we can write:

$$ \begin{aligned} &G_0(t)=0,\\ &G_1(t)=1,\\ &G_2(t)=2x-1,\\ &G_3(t)=3x^2-3x,\\ &G_i(t)=2B_i(t)-2^{i+1}B_i(t) \end{aligned} $$

The set G(t) = $G_0(t), G_1(t), \dots, G_n(t)$ is a complete orthogonal set in the Hilbert space $L^2[0, 1]$. Thus, we can expand any functions in this space in terms of G(t) polynomials [7,8]. The Genocchi polynomials satisfies in the following relations:

1. $$ \int_0^1 G_n(x)G_q(x)\,dx=\frac{2(-1)^n n!q!}{(n+q)!}G_{n+q} \quad , \quad q,n\geq 1, $$
2. $$ \frac{d}{dx}G_i(x)=iG_{i-1}(x), \qquad i\geq 1, $$

   and
3. $$ G_i(x+1)+G_i(x)=2ix^{i-1}. $$

The function approximation

Suppose G(t) is a N -vector as:

$$ G(t)=[G_1(t),G_2(t),\dots,G_N(t)]^T. \tag{7} $$

Function f (t) ∈ $L^2[0, 1]$ may be represented by the Genocchi polynomials as

$$ f(t)\approx\sum_{i=0}^{N}w_iG_i(t)=W^TG(t), \tag{8} $$

Where;

$$ W=[w_1,w_2,\dots,w_N]^T, $$

And, using Eq.(8) we obtain

$$ <f(t),G(t)>=<W^TG(t),G(t)>=W^TP, \tag{9} $$

Where;

$$ P=<G(t),G(t)>=\int_0^1 G(t)G^T(t)dx. \tag{10} $$

Thus,

$$ W=P^{-1}<f(t),G(t)>, \tag{11} $$

Operational matrices of fractional derivative

The kth derivative of G(x) is defined as:

$$ {G^{(k)}(x)}^T=G(x)({D_G}^T)^k, \tag{12} $$

Therefore, we can approximate derivative of arbitrary function f(t) using the Genocchi polynomials.

$$f'(t) = \sum_{i=0} w_i G_i'(t) = W^T D_G G(t), \tag{13}$$

Where w_i in Eq (13) is

$$w_i = \frac{1}{2i!}(f^{(i-1)}(1) + f^{(i-1)}(0)). \tag{14}$$

(i − 1) denotes the (i-1)th order derivative of f (t). In, for non-integer $\beta > 0$, the Caputo derivative of the vector G of order β can be defined as:

$${}_0^C D_t^\beta G(t) \approx D_\beta G(t), \tag{15}$$

Where D_β is a N × N matrix. It can be show:

$$D_\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum\limits_{r=\lceil\beta\rceil}^{\lceil\beta\rceil} \delta_{\lceil\beta\rceil,r,1} & \sum\limits_{r=\lceil\beta\rceil}^{\lceil\beta\rceil} \delta_{\lceil\beta\rceil,r,2} & \cdots & \sum\limits_{r=\lceil\beta\rceil}^{\lceil\beta\rceil} \delta_{\lceil\beta\rceil,r,N} \\ \sum\limits_{r=\lceil\beta\rceil}^{i} \delta_{i,r,1} & \sum\limits_{r=\lceil\beta\rceil}^{i} \delta_{i,r,2} & \cdots & \sum\limits_{r=\lceil\beta\rceil}^{i} \delta_{i,r,N} \\ \sum\limits_{r=\lceil\beta\rceil}^{N} \delta_{N,r,1} & \sum\limits_{r=\lceil\beta\rceil}^{N} \delta_{N,r,2} & & \sum\limits_{r=\lceil\beta\rceil}^{N} \delta_{N,r,N} \end{pmatrix}, \tag{16}$$

where $\eta_{i,k,j}$ is given by:

$$\delta_{i,r,j} = \frac{i! G_{i-r}}{(i-r)!\Gamma(r+1-\beta)} w_j. \tag{17}$$

Where G_{i-r} denotes the Genocchi numbers.

Sloving the fractional optimal control problems by the proposed method

In this section, we consider the FOCPs given in Eq (1)-(4). The factional state rate

$${}_0^C D_t^\beta x(t)$$

state t vector x(t) and control vector u(t), can be approximated by Genocchi polynomials as:

$$x_i(t) \approx G^T(t)E_i, \quad i = 1, ..., m \tag{18}$$
$$u_j(t) \approx G^T(t)U_j, \quad j = 1, ..., n \tag{19}$$
$${}_0^C D_t^\beta x(t) \approx G^T(t) {D_\beta}^T E_i, \;\; i = 1, ..., m. \tag{20}$$

Where D_β operational matrix given in Eq (16). We suppose:

$$\mathcal{G}_1 = I_n \otimes G, \tag{21}$$
$$\mathcal{G}_2 = I_m \otimes G, \tag{22}$$

I_m and I_n are m × m and n × n dimensional identity matrices, G(t) is N-vector, '⊗' denotes Kronecker product and $G_1(t)$ and $G_2(t)$ are matrices of order mN × m and nN × n.

Then using Eqs (21, 22), we can write:

$$x(t) \approx {\mathcal{G}_1}^T(t)E, \tag{23}$$
$$u(t) \approx {\mathcal{G}_2}^T(t)U, \tag{24}$$
$${}_0^C D_t^\beta x(t) \approx {\mathcal{G}_1}^T(t) {D_\beta}^T E. \tag{25}$$

where $E = [E_1^T, E_2^T, \ldots, E_m^T]^T$ and $U = [U_1^T, U_2^T, \ldots, U_n^T]^T$ are vectors of order mN and nN, respectively. By substituting Eqs.(23)-(25) in Eqs.(1)-(4), we get

$$J[E,U] = \int_0^1 h({\mathcal{G}_1}^T(t)E, {\mathcal{G}_2}^T(t)U, t)dt, \tag{26}$$
$${\mathcal{G}_1}^T(t) D_\beta^T E = f({\mathcal{G}_1}^T(t)E, {\mathcal{G}_2}^T(t)U, t), \tag{27}$$
$$g({\mathcal{G}_1}^T(t)E, {\mathcal{G}_2}^T(t)U, t) \leq 0, \tag{28}$$
$${\mathcal{G}_1}^T(0)E = x_0. \tag{29}$$

we assume two models for h in Eq (26):

h(x(t); u(t); t) is quadratic function. Thus Eq (26) is to form:

$$J[x,u] = \int_0^1 (x^T(t)Qx(t) + u^T(t)Ru(t))dt. \tag{30}$$

Where T denotes transposition, Q is positive semi-definite matrix and R is the positive

definite matrix. Using Eqs (23, 24) we get:

$$\begin{aligned} J[E,U] &\approx \int_0^1 (E^T \mathcal{G}_1(t) Q \mathcal{G}_1^T(t) E)dt + \int_0^1 (U^T \mathcal{G}_2(t) R \mathcal{G}_2^T(t) U)dt = \\ & E^T(\int_0^1 (\mathcal{G}_1(t) Q \mathcal{G}_1^T(t))dt)E + U^T(\int_0^1 (\mathcal{G}_2(t) R \mathcal{G}_2^T(t))dt)U, \end{aligned} \tag{31}$$

Therefore;

$$J[E,U] \approx E^T(\int_0^1 (\mathcal{G}_1(t)\mathcal{G}_1^T(t) \otimes Q(t))dt)E + U^T(\int_0^1 (\mathcal{G}_2(t)\mathcal{G}_2^T(t) \otimes R(t))dt)U, \tag{32}$$

Using by Eq (10) we get:

$$J[E,U] \approx E^T(P \otimes Q)E + U^T(P \otimes R)U. \tag{33}$$

h(x(t); u(t); t) is non-quadratic function. We evaluate objective function J by a suitable

Newton-Cots numerical integration as:

$$J[E,U] \approx \sum_{i=0}^{k} \rho_i h(\mathcal{G}_1^T(t_i)E, {\mathcal{G}^T}_2(t_i)U, t_i). \tag{34}$$

ρ_i, i = 0; 1; : : : ; k are the Newton-cots integration weight functions

By collocating Eq (27) at the points:

$$t_i = \frac{1}{2}(1 + \cos(\frac{(i-1)\pi}{N-1})), \;\; i = 1, 2, \ldots, N, \tag{35}$$

We get

$$\mathcal{G}_1^T(t_i) D_\beta{}^T E = f(\mathcal{G}_1^T(t_i)E, \mathcal{G}_2^T(t_i)U, t_i), \quad i = 1, 2, \ldots, N. \quad (36)$$

Now the problem changed to find the minimum solution of (33) or (34) with the conditions (29) and (36). Using lagrange multiplier method we have:

$$Min \quad J^*(E, U, \lambda, \mu) = J(E,U) + \sum_{i=1}^{N} \lambda_i \left(\mathcal{G}_1^T(t_i) D_\beta{}^T E - f(\mathcal{G}_1^T(t_i)E, \mathcal{G}_2^T(t_i)U, t_i)\right) + \mu(\mathcal{G}_1^T(0)E - x_0) \quad (37)$$

Where $\lambda=(\lambda_1, \lambda_2, \ldots, \lambda_N)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$ are the lagrange multipliers. To find the optimal solution of Eq (37) we put

$$\frac{\partial J^*[E,U,\lambda,\mu]}{\partial E} = 0, \quad \frac{\partial J^*[E,U,\lambda,\mu]}{\partial U} = 0, \quad \frac{\partial J^*[E,U,\lambda,\mu]}{\partial \lambda} = 0, \quad \frac{\partial J^*[E,U,\lambda,\mu]}{\partial \mu} = 0. \quad (38)$$

Eqs(38) given an algebraic system of equations which can be solved to find the values E, U, λ and μ [9-14].

Illustrative examples

Example 1 consider the following free final state FOCPs:

$$minimize \quad J = \int_0^1 [x_1^2(t) + x_2^2(t) + 0.005u^2(t)]dt \quad (39)$$

$$s.t. \quad {}_0^C D_t^\beta x_1(t) = x_2(t), \quad (40)$$

$${}_0^C D_t^\beta x_2(t) = -x_2(t) + u(t), \quad (41)$$

$$x_1(0) = 0, x_2(0) = -1, \quad (42)$$

$$x_2(t) \le 8(t-0.5)^2 - 0.5. \quad (43)$$

With ptimal value J*=0.171118. By applying present method, we obtain the numerical results (Table 1).

TABLE 1
Comparison of the value of J for β=1, for example 1

Methods	J
Classical chebyshev	
m=8, k=26	0.17358
m=16, k=28	0.17185
Hybrid function	
w=15, m=3, n=4	0.170136
w=15, m=4, n=4	0.170136
Haar wavelet collocation	
K=8	0.172548
k=16	0.171262
k=32	0.170112
Present method	
N=8	0.179078
N=10	0.170291
N=12	0.170049

values of N. Figure 1 shows the control function curves and plots of state vectors for β = 0.9, 0.99, 1 with N=10. Also, Table 1, shows the values of J for β=1 and gives a comparison between our results [15,16].

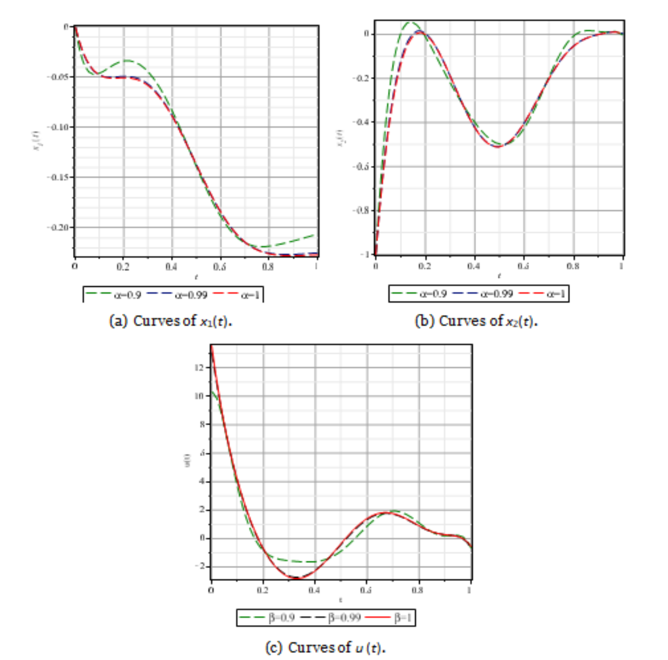


Figure 1) *Curves of* $x_1(t)$, $x_2(t)$ *and* $u(t)$ *with* N=10 *for*=0:9; 0:99

Example: This example has been chosen from. The problem is:

$$\begin{aligned} &\min \int_0^1 (-\ln(2))x(t)dt, \\ &s.t. \quad {}_0^C D_t^\beta x(t) = \ln(2)(x(t)+u(t)), \\ &\qquad |u(t)| \leq 1, \\ &\qquad x(t)+u(t) \leq 2, \\ &\qquad x(0)=0. \end{aligned} \tag{44}$$

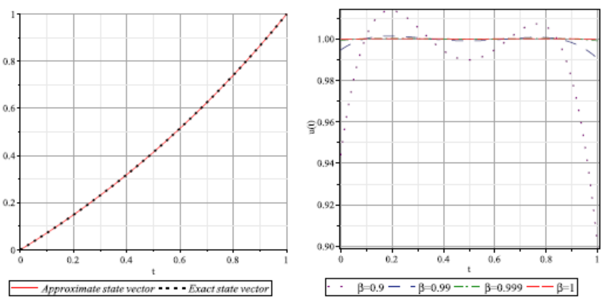
The state and control functions that minimize the preformance index J are given by $x^*(t)=2^t-1$ and $u^*(t)=1$, respectively. This problem for β is adapted from e and has been studied by several authors. This problem has the minimum value objective function $J^*=-0.30682$ for β=1. Table 2, gives the results reported and the presented method. Also, Figure 2, (a) shows the plot of x(t) for exact value of state vector and approximate state vector where β approach to 1 and (b) shows plot of control vector for the different values of β, that approaches to 1 [17-20]. Also, for this problem, we define error of x(t), $E_n(x)$, in the following from:

$$E_n(x) = \left(\frac{1}{n}\sum_{i=1}^{n}(x(t_i)-x_n(t_i))^2\right)^{\frac{1}{2}}$$

TABLE 2

Comparison of the value of J for β=1, for example 2

Methods	J	En(x)
Method		
n=2	-0.3064	8.07e-4
n=4	-0.30682	4.99e-5
n=8	-0.30685	3.09e-6
n=16	-0.30669	1.92e-7
n=32	-0.30685	1.20e-8
Method		
M=3, N=1	-0.30683	-
Method		
M=3, N=1	-0.30684	-
Method		
M=3	-0.30685	-
Present method		
N=5	-0.30685	9.61e-6
N=6	-0.30685	2.98e − 6
N=8	-0.30685	3.21e − 9



(a) Curves of $x(t)$. (b) Curves of $u(t)$.

Figure 2) *Curves of x(t) and u(t) with N=5*

CONCLUSION

We demonstrated how to apply the Genocchi polynomials in the approximation of FOCPs. The developed technique proved to give accurate and consistent results for both the state and control variables. Computed errors between our approximate solutions and the analytical solutions of specific problems were negligible, proving the accuracy of our suggested scheme.

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