RESEARCH

A proof of collatz conjecture

HUANG Qingxue

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ABSTRACT

In 1937, German mathematician L. Collatz proposed the following conjecture: for any positive integer, if it is even, divide it by 2, if it is odd, multiply it by 3 and add 1 to get an even number. Continuing with the above rule, the final result will be 1. This paper gives a proof of this conjecture.

Key words: *Number theory; Collatz conjecture*

INTRODUCTION

In 1937, German mathematician L. Collatz proposed the following conjecture that for a definite positive integer p, if p is even, divide it by 2,if p is odd, multiply it by 3 and add 1 to get an even number. Continuing with the above rule, the final result will be 1.

Collatz conjecture has been studied by many people and has long been regarded as an unsolved problem [1-3]. This paper gives a proof of this conjecture.

Preliminaries

Definition 2.1:

Starting from a positive integer p, the process with the rule in introduction is called a Collatz sequence. p is said to be successful if 1 is finally obtained. Otherwise, p is not successful.

Remark:

Any positive integer p can be written as that $p = 2^k(2m-1)$,where m is a positive integer and k is a positive integer or 0. When $k > 0$, p is even, and when $k = 0$, $p=2m-1$ is odd. If any odd number is successful, then since the even number $p = 2^k(2m-1)$ is divided by 2 k times to get the odd number 2m-1, the even number $p = 2^k(2m-1)$ is also successful.

In other words, to prove that Collatz conjecture holds, it is sufficient to show that any odd number is successful.

In this paper, the following discussion focuses on the Collatz sequence for odd numbers, and the even numbers in the Collatz sequence are omitted.

Example: For a positive integer p = 44, its Collatz sequence is that 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2,1, ... By removing all even numbers, it becomes that 11, 17, 13, 5, 1,1....We will omit the duplicate odd number 1 later on. It becomes that 11, 17, 13, 5, 1. That is, the odd number 11 is successful. This implies that odd 17, odd 13, odd 5, odd 1 are all successful. This also implies that even 2^k, even $2^k \times 5$, even $2^k \times 13$, even $2^k \times 17$, even $2^k \times 11$ are all successful, where k = 1, 2, 3, ...

Definition 2.2:

A sequence obtained by omitting all even numbers in a Collatz sequence starting with an odd number p is called a Collatz odd sequence. p is said to be successful if 1 is finally obtained. Otherwise, p is not successful.

Remark:

Obviously, whether an odd number p is successful, and Definition 2.1 and Definition 2.2 are equivalent

Definition 2.3:

For two odd numbers p and q in a Collatz odd sequence, if $3p + 1= 2^k q$,where $k > 0$ is a positive integer, then p is said to be a front odd number of q ,and q is said to be a back odd number of p.

In the previous example, because $11 \times 3 + 1 = 17 \times 2$, where k = 1, so 11 is a front odd number of 17, and 17 is a back odd number of 11. Similarly, because $17 \times 3 + 1 = 13 \times 2^2$, where k = 2, so 17 is a front odd number of 13, 13 is a back odd number of 17, etc.

Theorem 2.1:

If p is a front odd number of q, then 4p+1 is also a front odd number of q. Further if $3p+1 = 2^kq$, where $k > 0$ is a positive integer, then $3(4p+1)+1 = 2^{k+2}q$.

Proof:

Since p is a front odd number of q, there exists a positive integer $k > 0$, such that $3p+1 = 2^kq$. So $3(4p+1)+1 = 12p+4 = 4(3p+1) = 4 \times 2^kq= 2^{k+2}q$. QED

Consider the sequence $\{a_n\}$:$a_1, a_2, \ldots, a_n, \ldots$,where a_1 is an odd number, and $a_n = 4a_{n-1} + 1$,n = 2,3,4,....

Corollary 2.2:

For the above sequence $\{a_n\}$, if the odd number a_1 is a front odd number of the odd number p, then each term in the sequence $\{a_n\}$ is a

School of Mathematical Sciences, Zhejiang University, Hangzhou, China

Correspondence: HUANG Qingxue, School of Mathematical Sciences, Zhejiang University, Hangzhou, China, e-mail: qxhuang@zju.edu.cn

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front odd number of p. Further if $3a_1 + 1 = 2^k p$, then $3a_2 + 1 = 2^{k+2} p$,... ,$3a_n + 1 = 2^{k+2(n-1)}p$, where n = 2,3,4,....

Proof:
This is a direct corollary of Theorem 2.1.
QED

Theorem 2.3:
For any odd number p, there is a back odd number q, and q is the unique back odd number of p.

Proof:
According to the rule of generating a Collatz odd sequence, it is immediate. QED

Theorem 2.4:
In a Collatz odd sequence, if p is a front odd number of p, (of course, p is also a back odd number of p). Then it is only case that p = 1.

Proof:
Since p is a front odd number of p, there exists a positive integer k > 0, such that $3p+1 = 2^k p$. So $2^k p - 3p = p(2^k - 3) = 1$,where p is odd.

Case 1: k = 1. Then p(2-3) = -p = 1, which cannot be true.
Case 2: k > 1. Then $2^k - 3$ is a positive integer, and $p(2^k - 3) = 1$. This equation can be true, only if p = 1 and k = 2. QED

Bijections between odd subsets and sequence sets

This paragraph gives two classification methods of the odd number set, and gives a classification of sequence sets of the odd number set, and gives a bijection between odd subsets and sequence sets.

The odd number set can be divided into three subsets {4n-1|n = 1, 2, 3, ...}, {4n+1|n is 0 or even}= {8n+1|n = 0, 1, 2, 3, ...}, {4n+1|n is odd}. They are disjoint with each other.

Definition 3.1:
Write A ={4n-1|n = 1,2,3,...},B ={4n+1|n is 0 or even}= {8n+1|n = 0,1,2,3,...},C ={4n+1|n is odd}.

Definition 3.2:
(1)Write S_1 = {{a_n}|$a_1 \in$ A, and $a_n = 4a_{n-1}+1$,n = 2,3,4,...},where {a_n} is a sequence with the first term $a_1 \in$ A, and $a_n = 4a_{n-1}+1$,n = 2,3,4,....

(2)Write S_2 = {{a_n}|$a_1 \in$ B, and $a_n = 4a_{n-1}+1$,n = 2,3,4,...},where {a_n} is a sequence with the first term $a_1 \in$ B, and $a_n = 4a_{n-1}+1$,n=2,3,4,....

Examples: The first two sequences in S_1 are the following {a_n} and {b_n}, where {a_n }:3,13,53,..., (a_1 = 3); {b_n }:7,29,117,..., (b_1 = 7).

The first two sequences in S_2 are the following {a_n} and {b_n}, where {a_n }:1,5,21,85,..., (a_1 = 1); {b_n }:9,37,149,..., (b_1 = 9).

Theorem 3.1:
All terms (odd numbers) in all sequences of S_1 and S_2 contain all odd numbers. And each odd number must be in the only one of these sequences.

Proof:
Note that all odd numbers can be divided into three subsets A, B, C. Any odd number in A can be the first term a_1 of some definite sequence {a_n}in S_1.Similarly,any odd number in B can be the first term a_1 of some definite sequence {a_n}in S_2. So it is enough to prove that all terms of all sequences in S_1 and S_2 contain any odd number with the form 4n+1 (n is odd) in C, and that any odd number in C must be in the only one of these sequences.

Let $p = 4n_1 + 1 \in$ C, that is any definite odd number in C, where n_1 is a definite odd number. But all odd numbers can be divided into three subsets A, B, C.

Case 1: If $n_1 \in$ A, then n_1 is the first term a_1 of some definite sequence {a_n} in S_1.By the construction of the sequence {a_n} in S_1, $p = 4n_1 + 1$ is the second term a_2 of this definite sequence {a_n} in S_1.Similarly, if $n_1 \in$ B, then $p = 4n_1 + 1$ is the second term a_2 of some definite sequence {a_n} in S_2.

Case 2: If $n_1 \in$ C, then there exists a definite odd number n_2, such that $n_1 = 4n_2 + 1$. If $n_2 \in$ A (or $\in$ B), then according to Case 1, $n_1 = 4n_2 + 1$ is the second term a_2 of some definite sequence {a_n}in S_1 (or S_2), and $p = 4n_1 + 1$ is the third term a_3 of this definite sequence {a_n}in S_1 (or S_2).

Case 3: If $n_2 \in$ C, then we are back to the beginning of case 2. Continuing, since $p = 4n_1 + 1$ is a definite odd number in C, and the odd number 1 $\in$ B and 3 $\in$ A ,Case 2 cannot occur indefinitely. So, after finitely many case 2, we can always get some n_k , so that $n_{k-1} = 4n_k + 1$, $n_1 > n_2 > ... > n_k$, $n_1,n_2,...,n_{k-1} \in$ C and $n_k \in$ A (or $\in$ B), where n_k is the first term a_1 of some definite sequence {a_n}in S_1 (or S_2) ,n_{k-1} is the 2nd term a_2 of the sequence ,...,n_1 is the kth term a_k ,and $p = 4n_1 + 1$ is the k+1st term a_{k+1} of the definite sequence {a_n}in S_1 (or S_2). QED

Note that all odd numbers can again be divided into three subsets : {6n-3| n = 1,2,3,...}, {6n-1| n = 1,2,3,...}, {6n+1| n = 0,1,2,3,...}.

Definition 3.3: Write D = {6n-1| n = 1,2,3,...},E = {6n+1| n = 0,1,2,3,...},and F ={6n-3| n = 1,2,3,...}. Since 6n-3 is divisible by 3,an odd number in F is also called a triple odd number.

Lemma 3.2:
The odd number 4n-1 in A is a front odd number of the odd number 6n-1 in D, or 6n-1 is the back odd number of 4n-1, where n = 1, 2, 3,....

Proof:
Since 3(4n-1) + 1 = 2(6n-1), the result holds.
QED
Note that B = {4n+1|n is 0 or even} = {8n+1|n = 0,1,2,3,...}.

Lemma 3.3:
The odd number 8n+1 in B is a front odd number of the odd number 6n+1 in E, or 6n+1 is the back odd number of 8n+1, where n = 0,1,2,3,....

Proof:
Since $3(8n+1) + 1 = 2^2(6n+1)$, the result holds.
QED

Lemma 3.4:
The triple odd number 6n-3 in F cannot be a back odd number of any odd number, where n = 1, 2, 3,....

Proof:
Suppose that 6n-3 is a back odd number of some odd number p= 2m 1.Then $3(2m-1)+1 = 2^k(6n-3)$ holds for some positive integer k. Att his point, the right side $2^k(6n-3)$ of the equation is divisible by 3,w hile the left side 3(2m-1)+1 of the equation is not divisible by 3.C ontradiction.QED

Theorem 3.5:
There exists a bijection f_1: f_1（D） = S_1, such that for any 6k-1 $\in$ D,f_1(6k-1) = {a_n} = {a_n|a_1 = 4k-1,$a_n = 4a_{n-1} + 1$,n = 2,3,4,...}$\in S_1$,k =

1,2,3,..., where all front numbers of 6k-1 are exactly all terms of the sequence $\{a_n\}$,and $3a_n + 1 = 2^{2n-1}(6k-1)$ (n = 1,2,3,...);conversely, all terms of the sequence $\{a_n\}$ have the unique back number 6k-1.

Proof:
By Lemma 3.2,for a definite positive integer k, the first term 4k-1 of the sequence $\{a_n\}$ is a front odd number of the odd number 6k-1. Then by Corollary 2.2, each term of $\{a_n\}$ is a front odd number of 6k-1, and $3a_n + 1 = 2^{2n-1}(6k-1)$ (n = 1, 2, 3,...). We prove that any front odd numbers of 6k-1 is contained in this sequence $\{a_n\}$.

If an odd number p is not in $\{a_n\}$, then by theorem 3.1,p is a term of some other sequence $\{b_n\}$. We prove that p cannot be a front odd number of 6k-1.

Case 1: $\{b_n\}\in S_1$.Then the first term of $\{b_n\}$ is b_1,and $b_1 = 4s-1$,where s is a definite positive integer with $s \neq k$. As above, each term of $\{b_n\}$ is a front odd number of 6s-1. So, p is also a front odd number of 6s-1.By Theorem 2.3,6s-1 is the unique back odd number of p. Since $s \neq k$,6k-1 cannot be a back odd number of p. In other words, p cannot be a front odd number of 6k-1.

Case 2: $\{b_n\}\in S_2$.Then $b_1 = 8s+1$,where s is 0 or a definite positive integer. Then by Lemma 3.3,$b_1 = 8s+1$ is a front odd number of 6s+1.Then by Corollary 2.2, each term of $\{b_n\}$ is a front odd number of 6s+1.So,p is also a front odd number of 6s+1.Since $D \cap E = \emptyset$,$6k-1 \in D$ and $6s+1 \in E$,$6k-1 \neq 6s+1$.By Theorem 2.3, 6k-1 cannot be a back odd number of p. In other words, p cannot be a front odd number of 6k-1.

Therefore, any front odd numbers of 6k-1 is contained in this sequence $\{a_n\}$. So, for any odd number 6k-1 in D,$f_1(6k-1) = \{a_n | a_1 = 4k-1, a_n = 4a_{n-1}+1, n=2,3,4,...\}$,where k = 1,2,3,....

Conversely, by Theorem 2.3, for any sequence $\{a_n\} = \{a_n | a_1 = 4k-1, a_n = 4a_{n-1}+1, n = 2,3,4,...\}$ in S_1,all terms of the sequence $\{a_n\}$ have the unique back odd number 6k-1.So,$f_1^{-1}(\{a_n | a_1 = 4k-1, a_n = 4a_{n-1}+1, n=2,3,4,...\}) = 6k-1$,where k = 1,2,3,....

To sum up, we obtain that $f_1(D) = S_1$ is a bijection.
QED

Examples: Take k = 1,$f_1(5) = \{a_n\}$:3,13,53,213,.... Each term of $\{a_n\}$ is a front odd number of $5 \in D$. Take k = 2, $f_1(11) = \{a_n\}$:7,29,117,469,.... Each term of $\{a_n\}$ is a front odd number of $11 \in D$. etc.

Theorem 3.6:
There exists a bijection f_2: $f_2(E) = S_2$, such that for any $6k+1 \in E$,$f_2(6k+1) = \{a_n\} = \{a_n | a_1 = 8k+1, a_n = 4a_{n-1}+1, n=2,3,4,...\} \in S_2$,k = 1,2,3,..., where all front numbers of 6k+1 are exactly all terms of the sequence $\{a_n\}$,and $3a_n + 1 = 2^{2n}(6k+1)$ (n = 1,2,3,...);conversely, all terms of the sequence $\{a_n\}$ have the unique back number 6k+1.

Proof:
Following the method proved in Theorem 3.5, the result holds.
QED
Examples: Take k = 0, $f_2(1) = \{a_n\}$:1,5,21,85,....Each term of $\{a_n\}$ is a front odd number of $1 \in E$. Take k = 1, $f_2(7) = \{a_n\}$:9,37,149,597,....Each term of $\{a_n\}$ is a front odd number of $7 \in E$. etc.

Corollary 3.7: There exists a bijection f: $f(D \cup E) = S_1 \cup S_2$, where if an odd number $p \in D$, then define $f(p) = f_1(p)$,and if $p \in E$, then define $f(p) = f_2(p)$.
Proof:
Since $D \cap E = \emptyset$, and $S_1 \cap S_2 = \emptyset$, by Theorem 3.5 and Theorem 3.6, it is immediate. QED

Successful Odd Sequence Set H
This paragraph gives such an odd sequence set H that an odd number p is successful, if and only if p is in a sequence of H.
Theorem 4.1:
Let p be an odd number and q = 4p+1. Then

(1) if $p \in E$, then $q \in D$;
(2) if $p \in D$, then $q \in F$;
(3) if $p \in F$, then $q \in E$.

Proof:

(1) Since $p \in E$, let p = 6k+1, where k is 0 or a definite positive integer. Then q = 4p+1 = 4(6k+1)+1 = 24k+5 = 6(4k+1)-1, so $q \in D$.

(2) Since $p \in D$, let p = 6k-1,where k is a definite positive integer. Then q = 4p+1 = 4(6k-1)+1 = 24k-3 = 6(4k)-3, so $q \in F$.

(3) Since $p \in F$, let p = 6k-3, where k is a definite positivei nteger. Then q = 4p+1 = 4(6k-3)+1 = 24k-11= 6(4k-2)+1, s o $q \in E$. QED

Remarks: For a sequence $\{a_n\} \in S_1$, $a_1 = 4k-1 \in A$. And all odd numbers in A are {4k-1}:3, 7, 11, 15, 19, 23, ...; where 3, 15, ... ∈ F, and 7, 19, ... ∈ E, and 11, 23, ...∈ D.

For a sequence $\{a_n\}\in S_2$, $a_1 = 8k+1 \in B$. And all odd numbers in B are {8k+1}:1, 9, 17, 25, 33, 41 ,...; where 9, 33,...∈ F, and 1, 25,...∈ E, and 17, 41, ...∈ D.

That is, the first term a_1 can be any one in three odd subsets D, E, F, whether the sequence $\{a_n\} \in S_1$ or $\{a_n\}\in S_2$.

Theorem 4.2:
Let $\{a_n\} \in S_1$ or $\{a_n\} \in S_2$,

(1) if $a_1 \in E$, then $a_{3k-2} \in E$,$a_{3k-1} \in D$,$a_{3k} \in F$, where k = 1,2,3,...;
(2) if $a_1 \in F$, then $a_{3k-2} \in F$,$a_{3k-1} \in E$,$a_{3k} \in D$, where k = 1,2,3,...;
(3) if $a_1 \in D$, then $a_{3k-2} \in D$,$a_{3k-1} \in F$,$a_{3k} \in E$, where k = 1,2,3,....

Proof:
This is a direct corollary of Theorem 4.1.
QED
Remark:
According to Theorem 4.2, any term of a sequence $\{a_n\}$ in S_1 or S_2 can belong to any one in three odd subsets D,E,F.

Collatz sequences now under discussion are Collatz odd sequences, so the following definition is given.

Definition 4.1:
If $p_k, p_{k-1}, ..., p_1$,1 is a Collatz odd sequence with k odd numbers other than 1, then we say that p_k is k steps successful; and it is specified that the odd number 1 is 0 step successful.

Example: 11, 17, 13, 5, 1 is a Collatz odd sequence given in Preliminaries. Then we have: odd 5 is 1 step successful, 13 is 2 step s successful, 17 is 3 steps successful, and 11 is 4 steps successful.S pecially,1 is 0 step successful.

Corollary 4.3:
If p is a front odd number of q ,p ≠ 1 and q is k steps successful, then p is k+1 steps successful. Specially, p = 1 is 0 step successful.

Proof:
This is a direct consequence of Definition 4.1.
QED
Note that in the following, f_1, f_2 and f are the three bijections given in Theorem 3.5, Theorem 3.6 and Corollary 3.7, where f (D) = f_1 (D) = S_1, and f (E) = f_2 (E) = S_2. **For simplicity, if R is a sequence set, a sequence $\{a_n\} \in R$, and $a_i \in \{a_n\}$ is a term of $\{a_n\}$, then write $a_i \in R$. If $a_i \in R$ and $a_i \in D$, then write $a_i \in R \cap D$**, etc.

In the following, by a recursive method, we construct such a sequence set H that an odd number p is successful, if and only if p is in a sequence of H.

By Theorem 3.6 and Corollary 3.7, since $1 \in E$, $f(1) = f_2(1) = \{a_n\}$:1,5,21,85,341,…$\in S_2$.

Write $H_1 = \{f(1)\} = \{f_2(1)\} = \{\{a_n\}$:1,5,21,85,341,…$\}$, where H_1 is a sequence set, and there is exactly one sequence in H_1.

In this sequence of H_1, since $a_1 = 1 \in E$, by Theorem 4.2, $a_{3k-2} \in E$, $a_{3k-1} \in D$, $a_{3k} \in F$, where k = 1,2,3,…. By Lemma 3.4, a_{3k} ($\in F$) has no front odd number, where k = 1,2,3,…. By Theorem 3.5, for a definite a_{3k-1} ($\in D$), there is the unique odd sequence $\{b_n\} \in S_1$ such that $f_1(a_{3k-1}) = \{b_n\}$, where k = 1,2,3,…, and each term of $\{b_n\}$ is a front odd number of a_{3k-1}. By Theorem 3.6, for a definite a_{3k-2} ($\in E$), there is the unique odd sequence $\{c_n\} \in S_2$ such that $f_2(a_{3k-2}) = \{c_n\}$, where k = 1,2,3,…, and each term of $\{c_n\}$ is a front odd number of a_{3k-2}.

Write $H_2 = \{f_1(p) \mid p \in H_1 \cap D\} \cup \{f_2(p) \mid p \in H_1 \cap E$ and $p \neq 1\}$. Note that $(H_1 \cap D) \cup (H_1 \cap E) = H_1 \cap (D \cup E$. By Corollary 3.7, $H_2 = \{f(p) \mid p \in H_1 \cap (D \cup E$ and $p \neq 1\}$. (Note that the removal of $f(1) = f_2(1)$ in H_2 is to prevent the sequence that appeared in H_1 from reappearing in H_2.)

Write $H_3 = \{f_1(p) \mid p \in H_2 \cap D\} \cup \{f_2(p) \mid p \in H_2 \cap E\} = \{f(p) \mid p \in H_2 \cap (D \cup E)\}$.

Assume that the sequence set H_n has been formed.

Write $H_{n+1} = \{f(p) \mid p \in H_n \cap (D \cup E)\}$.

Now the sequence sets $H_1, H_2, H_3, \dots, H_n, \dots$ have been constructed.

Write $H = \bigcup_{n=1}^{\infty} H_n$.

Remark:
To get an intuitive sense of the composition of the set H, several sequences in H_1, H_2 and H_3 are given here according to Theorem 3.5 and Theorem 3.6.

The unique sequence in H_1 is $f(1) = f_2(1) = \{a_n\}$:1,5,21,85,341,1365,….

Note that $H_2 = \{f_1(p) \mid p \in H_1 \cap D\} \cup \{f_2(p) \mid p \in H_1 \cap E$ and $p \neq 1\}$. In the sequence $\{a_n\}$ of H_1, a_3 = 21 , a_6 = 1365,… ($\in F$) have no front odd number. Since p =5,341$\in H_1 \cap D$, by Theorem 3.5, $f_1(5) = \{b_n\}$:3,13,53,…$\in H_2$, $f_1(341) = \{b_n\}$:227,909,3637,…$\in H_2$. And since p = 85 $\in H_1 \cap E$ and p = 85 ≠ 1, by Theorem 3.6, $f_2(85) = \{b_n\}$:113,453,1813,…$\in H_2$. And so on.
And see H_3. $H_3 = \{f_1(p) \mid p \in H_2 \cap D\} \cup \{f_2(p) \mid p \in H_2 \cap E\}$. Because $f_1(5) = \{b_n\}$:3,13,53,…$\in H_2$, where 3 ($\in F$) has no front odd number, 13 $\in H_2 \cap E$, and 53 $\in H_2 \cap D$, so, by Theorem 3.6, $f_2(13) = \{c_n\}$: 17,69,277,…$\in H_3$, and by Theorem 3.5, $f_1(53) = \{c_n\}$: 35,141,565,…$\in H_3$. Because $f_1(341) = \{b_n\}$:227,909,3637,…$\in H_2$, where 909 ($\in F$) has no front odd number, 227 $\in H_2 \cap D$, and 3637 $\in H_2 \cap E$, so, by Theorem 3.5, $f_1(227) = \{c_n\}$: 151,605,2401,…$\in H_3$, and by Theorem 3.6, $f_2(3637) = \{c_n\}$: 4849,19397,77589,…$\in H_3$. Because $f_2(85) = \{b_n\}$:113,453, 1813,…$\in H_2$, where 453 ($\in F$) has no front odd number, 113 $\in H_2 \cap D$, and 1813 $\in H_2 \cap E$, so, by Theorem 3.5, $f_1(113) = \{c_n\}$: 75,301,1205,…$\in H_3$, and by Theorem 3.6, $f_2(1813) = \{c_n\}$: 2417,9669,38677,… $\in H_3$. And so on.

Theorem 4.4:

(1) There are no identical sequences in all sequences of $H = \bigcup_{n=1}^{\infty} H_n$. And no two different sequences in them have the same term.

(2) All terms of any sequence in H_n are n steps successful, where n = 1, 2, 3, …, except for the odd number 1 in H_1.

(3) If an odd number p is successful, then p must be in some sequence of the sequence set H.

Proof:
(1)Induction. Because $H_1 = \{f(1)\}$ and $H_2 = \{f(p) \mid p \in H_1 \cap (D \cup E)$ and $p \neq 1\}$, so $H_1 \cup H_2 = \{f(p) \mid p \in H_1 \cap (D \cup E)\}$.

Because $H_1 = \{f(1)\}$ is a sequence set = $\{\{a_n\}$:1,5,21,85,341,1365,…$\}$, and it has not the same term, so, there is no same odd number in $H_1 \cap (D \cup E)$. Since f is a bijection, all sequences in $H_1 \cup H_2$ are not identical to each other. Then since $H_1 \cup H_2 \subset S_1 \cup S_2$, by Theorem 3.1, no two different sequences in $H_1 \cup H_2$ have the same term.

Consider $H_3 = \{f(p) \mid p \in H_2 \cap (D \cup E)\}$. Then $H_1 \cup H_2 \cup H_3 = \{f(p) \mid p \in H_1 \cap (D \cup E)\} \cup \{f(p) \mid p \in H_2 \cap (D \cup E)\} = \{f(p) \mid p \in (H_1 \cup H_2) \cap (D \cup E)\}$. Since no two different sequences in $H_1 \cup H_2$ have the same term, there is not the same odd number in $(H_1 \cup H_2) \cap (D \cup E)$. Since f is a bijection, all sequences in $H_1 \cup H_2 \cup H_3$ are not identical to each other. By Theorem 3.1, no two different sequences in $H_1 \cup H_2 \cup H_3$ have the same term.

Suppose all sequences in $\bigcup_{k=1}^{n} H_k$ are different from each other, and no two different sequences have the same term.

Consider $H_{n+1} = \{f(p) \mid p \in H_n \cap (D \cup E)\}$. Then $\bigcup_{k=1}^{n+1} H_k = \{f(p) \mid p \in H_1 \cap (D \cup E)\} \cup \{f(p) \mid p \in H_2 \cap (D \cup E)\} \cup \dots \cup \{f(p) \mid p \in H_n \cap (D \cup E)\} = \{f(p) \mid p \in (\bigcup_{k=1}^{n} H_k) \cap (D \cup E)\}$. By the inductive assumption, no two different sequences in $\bigcup_{k=1}^{n} H_k$ have the same term. So, there is no same odd number in $(\bigcup_{k=1}^{n} H_k) \cap (D \cup E)$. Since f is a bijection, all sequences in $\bigcup_{k=1}^{n+1} H_k$ are not identical to each other. By Theorem 3.1, no two different sequences in $\bigcup_{k=1}^{n+1} H_k$ have the same term.

(2)Induction. First of all, all terms of the sequence in H1 are the front odd numbers of the odd number 1. By Definition 4.1, all terms of the sequence in H1 are 1 step successful, except for the odd number 1.

Consider any sequence $\{a_n\}$ in H_2. Then $H_2 = \{f(p) \mid p \in H_1 \cap (D \cup E)$ and $p \neq 1\}$. So, there is a definite odd number q, such that $q \in H_1 \cap (D \cup E)$, $q \neq 1$, and $f(q) = \{a_n\}$. Note that all terms of $\{a_n\}$ are all front numbers of q, and $q \in H_1$, $q \neq 1$ is 1 step successful. By Corollary 4.3, all terms of $\{a_n\}$ are 2 steps successful. Therefore, all terms of any sequence in H_2 are 2 steps successful.

Assume that all terms of any sequence in H_n are n steps successful, $n \geqq 2$.

Consider any sequence $\{b_n\}$ in H_{n+1}.Then $H_{n+1} = \{f(p) \mid p \in H_n \cap (D \cup E)\}$. So, there is a definite odd number q, such that $q \in H_n \cap (D \cup E)$, and $f(q) = \{b_n\}$. By the inductive assumption, $q \in H_n$ is n steps successful. Note that all terms of $\{b_n\}$ are all front numbers of q. By Corollary 4.3, all terms of $\{b_n\}$ are n+1 steps successful. Therefore, all terms of any sequence in H_{n+1} are n+1 steps successful.

(3) Firstly, the odd number p = 1 is 0 step successful,$1 \in H_1$.Now let $p \neq 1$.Since the odd number p is successful, there must exist some positive integer n, such that p is n steps successful. It is enough to prove that p must be in some sequence of H_n.

Induction. Look at n = 1.Let p be 1 step successful. Then $p \neq 1$.And let Collatz odd sequence of p be p,1. Then p is a front odd number of 1. Note that $H_1 = \{f(1)\} = \{\{a_n\}: 1, 5, 21, 85, 341, \ldots\}$,and f is a bijection, so, all front odd numbers of 1 belong to H_1.So,$p \in H_1$.

Look at n = 2.Let p be 2 steps successful. And let Collatz odd sequence of p be p ,p_1 ,1. Then the odd number p_1 is 1 step successful, and $p_1 \neq 1$.As above,$p_1 \in H_1$ and $p_1 \neq 1$.Because p_1 has the front odd number p, so $p_1 \in D \cup E$, and $p_1 \in H_1 \cap (D \cup E)$ and $p_1 \neq 1$. Note that $H_2 = \{f(q) \mid q \in H_1 \cap (D \cup E)$ and $q \neq 1\}$,then $f(p_1) \in H_2$,where $f(p_1)$ is a sequence of H_2.And $f(p_1)$ contain all front odd numbers of p_1 ,and p is a front odd number of p_1.So,$p \in f(p_1) \in H_2$.

Assume that the result holds for n = k, that is, if the odd number p is k steps successful, then $p \in H_k$.

Now let n = k+1, that is, the odd number p is k+1 steps successful. And let Collatz odd sequence of the odd number p be p ,p_k ,p_{k-1} ,…,p_1 ,1. Then the odd number p_k is k steps successful. By inductive assumption, $p_k \in H_k$. Because p_k has the front odd number p,so $p_k \in D \cup E$,and $p_k \in H_k \cap (D \cup E)$. Note that $H_{k+1} = \{f(q) \mid q \in H_k \cap (D \cup E)\}$,then $f(p_k) \in H_{k+1}$, where $f(p_k)$ is a sequence of H_{k+1}.And $f(p_k)$ contain all front odd numbers of p_k ,and p is a front odd number of p_k. So, $p \in f(p_k) \in H_{k+1}$.

Possibility of not successful odd sequence set

Remark:

From Theorem 4.4, it follows that every term of every sequence in $H = \bigcup_{n=1}^{\infty} H_n$ is successful, and every odd number that is successful must be in one of these sequences of H. Therefore, to prove that all odd numbers are successful, it is sufficient to show that every odd number is in one of these sequences of H.

Lemma 5.1:

(1) If p is a front odd number of q, and the odd number q is not successful, then the odd number p is also not successful.

(2) If r is a back odd number of q, and the odd number q is not successful, then the odd number r is also not successful.

Proof:

(1) First, since q is not successful, $q \neq 1$. Assume that the odd number p is successful, then there exists a positive integer k, such that p is k steps successful. And by Theorem 2.3, q is the unique back odd number of p. Then, by Definition 4.1, k > 1, and q is k-1 steps successful. Contradiction.

(2) Assume that the odd number r is successful, then there exists a positive integer m, such that r is m steps successful, so q is m+1 steps successful. Contradiction.
QED

By Theorem 3.1, any odd number is a term of only one sequence in $S_1 \cup S_2$.

Lemma 5.2:

If an odd number p is not successful, and p is a term of some sequence $\{a_n\}$ in $S_1 \cup S_2$,then all terms of $\{a_n\}$ are not successful.

Proof:

From Theorem 4.4, it follows that all odd numbers in the sequence set H are successful. Since the odd number p is not successful, p is not in H. And the odd number p is a term in some sequence $\{a_n\}$.Since H ist he sequence set , that sequence $\{a_n\}$ is also not in H. In the otherw ords, all terms of $\{a_n\}$ are not in H. By Theorem 4.4, all terms of $\{a_n\}$ar e not successful. QED

Remarks:

(1)In a Collatz odd sequence, if there is a situation like $q_1, q_2, \ldots, q_k, q_1$, where $q_1, q_2, \ldots, q_k$ are different from each other, and the odd number q_1 recurring, then we call these k odd numbers to form a k-cycle. In fact, it is impossible to have such a k-cycle in a successful Collatz odd sequence. Because there must be a positive integer s for a successful odd number q_1,so that q_1 is s steps successful, so, there is a Collatz odd sequence $q_1, p_{s-1}, p_{s-2}, \ldots, p_1, 1$. Assume q_1 is in a k-cycle $q_1, q_2, \ldots, q_k, q_1$.Then q_1 has a Collatz odd sequence $q_1, q_2, \ldots, q_r, p_t, p_{t-1}, \ldots, p_1, 1$,where $q_2 = p_{s-1}, \ldots, q_r = p_{t+1}$, $1 \leqq r \leqq k$, r is a definite integer. Then q_r has 2 back odd numbers q_{r+1} and p_t, where if r = k, then $q_{r+1} = q_{k+1} = q_1$,and if r = 1,then $p_t = p_{s-1}$.By Theorem 2.3,this is impossible.

(2)Assuming that p_1 is not successful, and p_1 is in a k-cycle $p_1, p_2, \ldots, p_k, p_1$.Then by Theorem 2.3,Collatz odd sequence of these k odd numbers is the k-cycle, and any one of these k odd numbers can not appear in other j-cycle. And these k odd numbers are not successful.

(3)Let p be a front odd number of q. Then there exists a positive integer i, such that $3p+1 = 2^i q$.If i = 1, then $q = (3p+1)/2 > p$. If $i \geqq 2$,then $q = (3p+1)/2^i < p$. Note that any odd number is a term of some sequence in $S_1 \cup S_2$.By Theorem 3.5,for any sequence $\{a_n\} = \{a_n | a_1 = 4k-1, a_n = 4a_{n-1}+1, n=2,3,4,\ldots\} \in S_1$,$3a_n + 1 = 2^{2n-1}(6k-1)$ (n = 1,2,3,…),where any term of $\{a_n\}$ is a front odd number of 6k-1.And when n =1,that is, i = 2n-1 = 1,$6k-1 > a_1$.When $n \geqq 2$,that is, $i = 2n-1 \geqq 3$,$6k-1 < a_n$. And by Theorem 3.6,for any sequence $\{a_n\} = \{a_n | a_1 = 8k+1, a_n = 4a_{n-1}+1, n=2,3,4,\ldots\} \in S_2$,$3a_n + 1 = 2^{2n}(6k+1)$ (n = 1,2,3,…),where any term of $\{a_n\}$ is a front odd number of 6k+1.When n = 1,2,3,…,that is, $i = 2n \geqq 2$,$6k+1 < a_n$.

Lemma 5.3:

(1)There are no 2-cycles in Collatz odd sequences.

(2)There are no 3-cycles in Collatz odd sequences.

Proof:

(1)Suppose that there is a 2-cycle q_1, q_2, q_1 in Collatz odd sequences. Let $q_1 < q_2$.Because q_1 is a front odd number of q_2,so there is a positive integer i, such that $3q_1+1 = 2^i q_2$.By Remark(3) above, i = 1,that is,$3q_1+1 = 2q_2$.And because q_2 is a front odd number of q_1,so there is a positive integer j, such that $3q_2+1 = 2^j q_1$,i.e,$(3q_2+1)/2^j = q_1$.By Remark(3) above, since $q_2 > q_1$,$j \geqq 2$.But $(3q_1+1)/2 = q_2$.When j = 2,$(3q_2+1)/2^j = (3((3q_1+1)/2)+1)/2^2 = (9q_1+5)/8 > q_1$.When $j \geqq 3$,$(3q_2+1)/2^j \leqq (3((3q_1+1)/2)+1)/2^3 = (9q_1+5)/16 < q_1$. That is to say, $(3q_2+1)/2^j \neq q_1$.So, there are no 2-cycles in Collatz odd sequences.

(2)And suppose that there is a 3-cycle q_1, q_2, q_3, q_1 in Collatz odd sequences. Let $q_1 < q_2$ and $q_1 < q_3$.The following are divided into two cases.

Case 1：

$q_2 > q_3$.Since $q_1 < q_2$,and q_1 is a front odd number of q_2,by Remark(3) above,$3q_1+1 = 2q_2$,i.e,$(3q_1+1)/2 = q_2$.Since $q_2 > q_3$,and q_2 is a front odd number of q_3,$3q_2+1 = 2^i q_3$,i.e,$(3q_2+1)/2^i = q_3$,where $i \geqq 2$.Since $q_3 > q_1$,and q_3 is a front odd number of q_1,$3q_3+1 = 2^j q_1$,i.e,$(3q_3+1)/2^j = q_1$,where $j \geqq 2$.But $(3q_3+1)/2^j \leqq (3q_3+1)/2^2 = (3((3q_2+1)/2^i)+1)/2^2 \leqq (3((3q_2+1)/2^2)+1)/2^2 = (9q_2+7)/16 = (9((3q_1+1)/2)+7)/16 = (27q_1+23)/32 < q_1$.(Note that because small odd numbers must be successful, and odd numbers in k-cycle are all not successful, so, we can set $q_1 > 99$.) That is to say,$(3q_3+1)/2^j \neq q_1$. Contradiction.

Case 2：

$q_2 < q_3$.Since $q_1 < q_2$,and q_1 is a front odd number of q_2,by Remark(3) above,$3q_1+1 = 2q_2$,i.e,$(3q_1+1)/2 = q_2$.Since $q_2 < q_3$,and q_2 is a front odd number of q_3,$3q_2+1 = 2q_3$,i.e,$(3q_2+1)/2 = q_3$.Since $q_3 > q_1$,and q_3 is a front odd number of q_1,$3q_3+1 = 2^i q_1$,i.e,$(3q_3+1)/2^i = q_1$,where $i \geqq 2$.But $(3q_3+1)/2^i = (3((3q_2+1)/2)+1)/2^i = (9q_2+5)/2^{i+1} = (9((3q_1+1)/2)+5)/2^{i+1} = (27q_1+19)/2^{i+2}$, where $i \geqq 2$.And when $i = 2$,$(3q_3+1)/2^i = (27q_1+19)/2^{i+2} = (27q_1+19)/16 > q_1$.when $i \geqq 3$,$(3q_3+1)/2^i = (27q_1+19)/2^{i+2} \leqq (27q_1+19)/32 < q_1$,where set $q_1 > 99$. That is to say,$(3q_3+1)/2^i \neq q_1$. Contradiction.

So, there are no 3-cycles in Collatz odd sequences.

QED

Remark:

The same method can be used to prove that there are no 4-cycles,5-cycles,etc. But there are too many cases to consider, and it will not go on. Later, in Theorem 5.5, we will prove that the k-cycle does not exist.

Remark:

Suppose that an odd number q_1 is not successful. Then a Collatz odd sequence $q_1, q_2, \ldots, q_n, \ldots$ can be obtained, where q_{i+1} is the unique back odd number of q_i,$i = 1,2,3,\ldots$.By Lemma 5.1,each odd number in the Collatz odd sequence is not successful. In this point, there exist two cases for the Collatz odd sequence $q_1, q_2, \ldots, q_n, \ldots$

Case 1:

All odd numbers of the Collatz odd sequence $q_1, q_2, \ldots, q_n, \ldots$ are different from each other.

Case 2:

$q_1,q_2,\ldots,q_k,q_{k+1},q_{k+2}$ are different from each other, and q_{k+3} is an odd number in $q_1,q_2,\ldots,q_{k+2}$,such as,$q_{k+3} = q_3$.Then $q_3,q_4,\ldots,q_{k+2},q_3$ is a k-cycle, $k \geqq 4$.($k \geqq 4$ is because the non-existence of 2-cycle and 3-cycle has been proved in Lemma 5.3.).

And the Collatz odd sequence $q_1,q_2,\ldots,q_n,\ldots$ is changed to $q_1,q_2,q_3,\ldots,q_k,q_{k+1},q_{k+2},q_3,\ldots,q_k,q_{k+1},q_{k+2},q_3,\ldots$,by Theorem 2.3,here this k-cycle keeps appearing repeatedly. For convenience, in the following, the Collatz odd sequence has been replaced with $r_1,r_2,q_1,q_2,q_3,\ldots,q_k,q_1,q_2,q_3,\ldots,q_k,q_1,\ldots$.

Now, As with the construction of the sequence set H, from not successful Collatz odd sequence $q_1, q_2, \ldots, q_n, \ldots$, the following series G_n of sequence sets is constructed. In the above Case 2,$q_1,q_2,\ldots,q_n,\ldots$has been replaced with $q_1,q_2,q_3,\ldots,q_k,q_1,q_2,q_3,\ldots,q_k,q_1,\ldots$.

The following construction is unified for Case 1 and Case 2.In the construction of G_n, for Case 2,when $n = ik + m$, let $q_n = q_m$, where $i = 0,1,2,\ldots,m = 1,2,\ldots,k$.

Firstly, we construct a sequence set G_1.

Note that each odd number in the Collatz odd sequence $q_1, q_2,\ldots,q_n,\ldots$is not successful, and q_2 is the unique back odd number of q_1 .

Write $G_{11} = \{f(q_2)\} = \{\{a_n\}\}$,where $\{a_n\} \in S_1 \cup S_2$,all terms of $\{a_n\}$ are all front odd numbers of q_2,and the odd number q_1 is a term of $\{a_n\}$.

Write $G_{12} = \{f(p) \mid p \in G_{11} \cap (D \cup E)\}$.
Write $G_{13} = \{f(p) \mid p \in G_{12} \cap (D \cup E)\}$.

Assume that the sequence set G_{1n} has been formed.
Write $G_{1,n+1} = \{f(p) \mid p \in G_{1n} \cap (D \cup E)\}$.
Denote $G_1 = \bigcup_{n=1}^{\infty} G_{1n}$.

And construct a sequence set G_2.
Note that q_3 is the unique back odd number of q_2.
Write $G_{21} = \{f(q_3)\} = \{\{b_n\}\}$,where the odd number q_2 is a term of $\{b_n\}$.

Write $G_{22} = \{f(p) \mid p \in G_{21} \cap (D \cup E)\}$.
Write $G_{23} = \{f(p) \mid p \in G_{22} \cap (D \cup E)\}$.

Assume that the sequence set G_{2n} has been formed.
Write $G_{2,n+1} = \{f(p) \mid p \in G_{2n} \cap (D \cup E)\}$.
Denote $G_2 = \bigcup_{n=1}^{\infty} G_{2n}$.
So on and so forth.

Suppose again that $G_n = \bigcup_{k=1}^{\infty} G_{nk}$ has been constructed. In the construction of G_n, for Case 2,when $n = ik + m$, let $q_n = q_m$, where $i = 0,1,2,\ldots,m = 1,2,\ldots,k$.

At this point, G_{n1} has only one sequence, denoted $\{c_n\}$, and $q_n \in \{c_n\}$.q_{n+1} is the unique back odd number of q_n.

Write $G_{n+1,1} = \{f(q_{n+2})\} = \{\{d_n\}\}$.where the odd number q_{n+1} is a term of $\{d_n\}$.

Write $G_{n+1,2} = \{f(p) \mid p \in G_{n+1,1} \cap (D \cup E)\}$.
Write $G_{n+1,3} = \{f(p) \mid p \in G_{n+1,2} \cap (D \cup E)\}$.

Assume that the sequence set $G_{n+1,k}$ has been formed.
Write $G_{n+1,k+1} = \{f(p) \mid p \in G_{n+1,k} \cap (D \cup E)\}$.
Denote $G_{n+1} = \bigcup_{k=1}^{\infty} G_{n+1,k}$.

Now G_n has been constructed recursively, where $n = 1, 2, 3,\ldots$.

Remark:

For r_1 and r_2 mentioned in Case 2,because r_2 is a front odd number of q_1,and r_1 is a front odd number of r_2,and $q_1 \in G_{11}$,so,$r_2 \in f(q_1) \in G_{12}$ and $r_1 \in f(r_2) \in G_{13}$.In other words, r_1 and r_2 are all in G_1.

.The following Theorem 5.4 is valid for both Case 1 and Case 2.

Theorem 5.4:

(1) Any term of all sequences in the set G_n is not successful, where $n = 1, 2, 3,\ldots$.

(2) $G_1 \subset G_2 \subset \ldots \subset G_n \subset \ldots$, where G_i is a proper subset of G_{i+1},$i = 1,2,3,\ldots$.

Proof:

(1)Induction for G_{nk}, where $n = 1, 2, 3,\ldots$. Firstly, consider G_{n1}.Since q_n is not successful, and $q_n \in \{c_n\}$,(see G_{n1}.) by Lemma 5.2,any term of the unique sequence $\{c_n\}$ in G_{n1} is not successful. The result holds for $k = 1$.

Consider $G_{n2} = \{f(p) \mid p \in G_{n1} \cap (D \cup E)\}$.Let $f(q) = \{b_n\}(\in S_1 \cup S_2)$ be any sequence in G_{n2}.Then $q \in G_{n1} \cap (D \cup E)$,and all term of $\{b_n\}$ are all front odd numbers of q. Since $q \in G_{n1}$,q is not successful. And by Lemma 5.1(1), all term of $\{b_n\}$ are not successful. So the result holds when k = 2.

Assume that the result holds for k = i, that is, any term of any sequence in G_{ni} is not successful.

Consider $G_{n,i+1} = \{f(p) \mid p \in G_{ni} \cap (D \cup E)\}$.Let $f(q) = \{d_n\}(\in S_1 \cup S_2)$ be any sequence in $G_{n,i+1}$. Then $q \in G_{ni} \cap (D \cup E)$, and all term of $\{d_n\}$ are all front odd numbers of q. Since $q \in G_{ni}$, by inductive assumption, q is not successful. And by Lemma 5.1(1), all term of $\{d_n\}$ are not successful. So the result holds when k = i+1.

(2) First prove that $G_1 \subset G_2$. Look back at the construction of G_2. Note that $G_{11} = \{f(q_2)\} = \{\{a_n\}\}$, where $\{a_n\} \in S_1 \cup S_2$, and the odd number q_1 is a term of $\{a_n\}$. $G_{21} = \{f(q_3)\} = \{\{b_n\}\}$, where $\{b_n\} \in S_1 \cup S_2$,and the odd number q_2 is a term of $\{b_n\}$. $G_{22} = \{f(p) \mid p \in G_{21} \cap (D \cup E)\}$. Since $q_2 \in \{b_n\} \in G_{21}$, and $f(q_2) = \{a_n\}$, (any term of $\{a_n\}$ is a front odd numbers of q_2.)$q_2 \in G_{21} \cap (D \cup E)$.Thus $f(q_2) \in G_{22}$,i.e, $G_{11} \subset G_{22}$.Because $G_{12} = \{f(p) \mid p \in G_{11} \cap (D \cup E)\}$,and $G_{23} = \{f(p) \mid p \in G_{22} \cap (D \cup E)\}$, and $(G_{11} \cap (D \cup E)) \subset (G_{22} \cap (D \cup E))$, so, $G_{12} \subset G_{23}$. By analogy, it follows that $G_{13} \subset G_{24},\ldots,G_{1n} \subset G_{2,n+1}$, etc.

Since $G_1 = G_{11} \cup G_{12} \cup \ldots \cup G_{1n} \cup \ldots$
$G_2 = G_{21} \cup G_{22} \cup \ldots \cup G_{2n} \cup G_{2,n+1} \cup \ldots$
So $G_1 \subset G_2$.Obviously, G_1 is a proper subset of G_2. By analogy, $G_2 \subset G_3 \subset \ldots \subset G_n \subset \ldots$. QED

Theorem 5.5:

Case 2 cannot exist, that is, there are no k-cycles in not successful Collatz odd sequence.

Proof:

Firstly, for Case 2,by Theorem 5.4,$G_1 \subset G_2 \subset \ldots \subset G_{k+1}$,where G_i is a proper subset of G_{i+1},i = 1,2,...,k.So,G_1 is a proper subset of G_{k+1},

On the other hand, consider the constructions of G_{k+1} and G_1.Note that $G_{k+1,1} = \{f(q_{k+2})\}$, where $q_{k+2} = q_2$.So,$G_{k+1,1} = \{f(q_2)\}$.But $G_{11} = \{f(q_2)\}$,so,$G_{k+1,1} = G_{11}$.And because $G_{k+1,2} = \{f(p) \mid p \in G_{k+1,1} \cap (D \cup E)\} = \{f(p) \mid p \in G_{11} \cap (D \cup E)\}$,and $G_{12} = \{f(p) \mid p \in G_{11} \cap (D \cup E)\}$, so,$G_{k+1,2} = G_{12}$.By analogy,$G_{k+1,i} = G_{1i}$,where i = 3,4,5,....Thus,$G_1 = \bigcup_{n=1}^{\infty} G_{1n} = \bigcup_{n=1}^{\infty} G_{k+1,n} = G_{k+1}$. Contradiction. Case 2 cannot exist.

Next, we just need to discuss Case 1.Consider G_1 first.

Theorem 5.6:

If $q_1, q_2,\ldots, q_n,\ldots$is the Collatz odd sequence in the construction of G_n,where all odd numbers of the Collatz odd sequence are different from each other, then

(1) $q_2 \notin G_{1i}$,where i = 1,2,3,....
(2) There are no identical sequences in the sequence set $\bigcup_{i=1}^{\infty} G_{1i}$, and no two different sequences in $\bigcup_{i=1}^{\infty} G_{1i}$ have the same term.

Proof:

(1) Firstly,$G_{11} = \{f(q_2)\} = \{\{a_n\}\}$,where all terms of $\{a_n\}$ are all front odd numbers of q_2.Because q_2 is not successful,so,$q_2 \neq 1$, and by Theorem 2.4,$q_2 \notin G_{11}$.Suppose that q_2 is a term of some sequence $\{d_n\}$ in some G_{1i},where $i \geqq 2$ is a definite integer. Since $G_{1i} = \{f(p) \mid p \in G_{1,i-1} \cap (D \cup E)\}$,there exists a definite odd number $s \in G_{1,i-1} \cap (D \cup E)$,such that $f(s) = \{d_n\}$,and all terms of $\{d_n\}$ are all front odd numbers of s.But q_3 is the unique back odd number of q_2 $(\in\{d_n\})$,so,$q_3 = s \in G_{1,i-1}$.By analogy,$q_4 \in G_{1,i-2},\ldots,q_i \in G_{12},q_{i+1} \in G_{11}$.But $G_{11} = \{f(q_2)\} = \{\{a_n\}\}$,where all terms of $\{a_n\}$ are all front odd numbers of q_2.So,q_2 is a back odd number of $q_{i+1}(\in \{a_n\})$.Then q_{i+1} has two back odd number $q_2,q_{i+2}(\neq q_2)$,where $i \geqq 2$.By Theorem 2.3,this is impossible.

(2) Induction. First prove that the result holds for $G_{11} \cup G_{12}$.Note that $G_{11} = \{f(q_2)\} = \{\{a_n\}\}= \{f(p) \mid p \in \{q_2\} \cap (D \cup E)\}$,and $G_{12} = \{f(p) \mid p \in G_{11} \cap (D \cup E)\}$.So,$G_{11} \cup G_{12} = \{f(p) \mid p \in (\{q_2\} \cup G_{11}) \cap (D \cup E)\}$.By (1),$q_2 \notin G_{11}$.And $(\{q_2\} \cup G_{11}) \cap (D \cup E)$ has not the same odd number. Because f is a bijection, so, there are no identical sequences in the sequence set $G_{11} \cup G_{12}$. Then since $G_{11} \cup G_{12} \subset S_1 \cup S_2$,by Theorem 3.1, no two different sequences in $G_{11} \cup G_{12}$ have the same term.

Consider $G_{13} = \{f(p) \mid p \in G_{12} \cap (D \cup E)\}$.Then $\bigcup_{i=1}^{3} G_{1i} = \{f(p) \mid p \in (\{q_2\} \cup G_{11}) \cap (D \cup E)\} \cup \{f(p) \mid p \in G_{12} \cap (D \cup E)\} = \{f(p) \mid p \in (\{q_2\} \cup G_{11} \cup G_{12}) \cap (D \cup E)\}$.By (1),$q_2 \notin G_{11} \cup G_{12}$.And because there is no same odd number in $G_{11} \cup G_{12}$,so, there is no same odd number in $(\{q_2\} \cup G_{11} \cup G_{12}) \cap (D \cup E)$.Because f is a bijection, so, there are no identical sequences in the sequence set $\bigcup_{i=1}^{3} G_{1i}$. Then by Theorem 3.1, no two different sequences in $\bigcup_{i=1}^{3} G_{1i}$ have the same term.

Assume there are no identical sequences in the sequence set $\bigcup_{i=1}^{n} G_{1i}$, and no two different sequences in $\bigcup_{i=1}^{n} G_{1i}$ have the same term, i,e. there is no same odd number in $\bigcup_{i=1}^{n} G_{1i}$.

Note that $\bigcup_{i=1}^{n+1} G_{1i} = \{f(p) \mid p \in (\{q_2\} \cup (\bigcup_{i=1}^{n} G_{1i})) \cap (D \cup E)\}$.By (1),$q_2 \notin \bigcup_{i=1}^{n} G_{1i}$. By the inductive assumption, there is no same odd number in $\bigcup_{i=1}^{n} G_{1i}$. So, there is no same odd number in $(\{q_2\} \cup (\bigcup_{i=1}^{n} G_{1i})) \cap (D \cup E)$.Because f is a bijection, so, there are no identical sequences in the sequence set $\bigcup_{i=1}^{n+1} G_{1i}$ .Then by Theorem 3.1,no two different sequences in $\bigcup_{i=1}^{n+1} G_{1i}$ have the same term.
QED

The same method can be used to prove the following Theorem 5.7.

Theorem 5.7:

If $q_1, q_2,\ldots,q_n,\ldots$is the Collatz odd sequence in the construction of G_n, where all odd numbers of the Collatz odd sequence are different from each other, then for n = 2,3,4,...,

(1) $q_{n+1} \notin G_{ni}$, where i = 1,2,3,....
(2) There are no identical sequences in the sequence set $\bigcup_{i=1}^{\infty} G_{ni}$, and no two different sequences in $\bigcup_{i=1}^{\infty} G_{ni}$ have the same term.
QED

RESULTS

In order to make a comparison between the sequence sets H and G_n,, another construction method of H is introduced here.

Let the odd number q_1 be k steps successful. And let $q_1, q_2,\ldots,q_k$,1 (here set $q_{k+1} = 1$)be the Collatz odd sequence.

Next the following series T_i of sequence sets is constructed, where i = 1,2,...,k.

Firstly, we construct a sequence set T_1.

Note that each odd number in the Collatz odd sequence $q_1,q_2,\ldots,q_k$,1 is successful, and q_2 is the unique back odd number of q_1 .

Write $T_{11} = \{f(q_2)\} = \{\{a_n\}\}$,where $\{a_n\} \in S_1 \cup S_2$,and the odd number q_1 is a term of $\{a_n\}$.

Write $T_{12} = \{f(p) \mid p \in T_{11} \cap (D \cup E)\}$.Assume that the sequence set T_{1n} has been formed.

Write $T_{1,n+1} = \{f(p) \mid p \in T_{1n} \cap (D \cup E)\}$.
Denote $T_1 = \bigcup_{n=1}^{\infty} T_{1n}$.

Note that q_3 is the unique back odd number of q_2.
Write $T_{21} = \{f(q_3)\} = \{\{b_n\}\}$,where $\{b_n\} \in S_1 \cup S_2$,and the odd number q_2 is a term of $\{b_n\}$.

Write $T_{22} = \{f(p) \mid p \in T_{21} \cap (D \cup E)\}$.
Assume that the sequence set T_{2n} has been formed.
Write $T_{2,n+1} = \{f(p) \mid p \in T_{2n} \cap (D \cup E)\}$.
Denote $T_2 = \bigcup_{n=1}^{\infty} T_{2n}$.
So on and so forth.
Suppose again that $T_i = \bigcup_{n=1}^{\infty} T_{in}$ has been constructed,where i = 3,4,...,k-2.

At this point, T_{i1} has only one sequence, denoted $\{c_n\}$,and $q_i \in \{c_n\}$.q_{i+1} is the unique back odd number of q_i.
Write $T_{i+1,1} = \{f(q_{i+2})\} = \{\{d_n\}\}$,where $\{d_n\} \in S_1 \cup S_2$,and the odd number q_{i+1} is a term of $\{d_n\}$.
Write $T_{i+1,2} = \{f(p) \mid p \in T_{i+1,1} \cap (D \cup E)\}$.
Assume that the sequence set $T_{i+1,j}$ has been formed.
Write $T_{i+1,j+1} = \{f(p) \mid p \in T_{i+1,j} \cap (D \cup E)\}$.
Denote $T_{i+1} = \bigcup_{j=1}^{\infty} T_{i+1,j}$.

And consider i = k-1.Then $q_{i+2} = q_{k+1} = 1$,$T_{i+1,1} = T_{k1} = \{f(q_{i+2})\} = \{f(1)\}$,where $f(1)$ is the odd sequence $\{e_n\}$:1,5,21,85,341,...,$q_{i+1} = q_k$ is a term of $\{e_n\}$.
Write $T_{k2} = \{f(p) \mid p \in T_{k1} \cap (D \cup E)$,and $p \neq 1\}$.
Assume that the sequence set T_{kj} has been formed, where $j \geqq 2$.
Write $T_{k,j+1} = \{f(p) \mid p \in T_{kj} \cap (D \cup E)\}$.
Denote $T_k = \bigcup_{j=1}^{\infty} T_{kj}$.

Remarks:
(1)Note that T_k = H, that is, the sequence set H in §5 can also bec onstructed by the above method, and this process is the same as theg eneration process of G_n.
(2)See §5.Consider the constructions of G_1 and G_2.$G_{11} = \{f(q_2)\}$,and $G_{21} = \{f(q_3)\} = \{\{b_n\}\}$,where the odd number q_2 is just a general odd number in the sequence $\{b_n\}$ of G_{21}, or in $G_{21} \cap (D \cup E)$.$G_{11} = \{f(q_2)\}$ grows in the front odd number direction (let's say) to obtain a G_1. While there are infinitely many odd numbers in $G_{21} \cap (D \cup E)$,and G_2 is obtained by growing the infinitely many odd numbers of $G_{21} \cap (D \cup E)$ in the front odd number direction. Where each odd number in $G_{21} \cap (D \cup E)$ also generates a sequence set equivalent to G_1.Thus,G_2 is "infinitely many times" larger than G_1 .

Similarly, for n = 3,4,...,G_n is "infinitely many times" larger than G_{n-1}. It can also be obtained for i = 1,2,...,k-1,T_{i+1} is "infinitely many times" larger than T_i.

(3)By the constructions of G_n and T_i, the difference between sequence set G_n and the sequence set T_i is that all terms of all sequences in T_i are successful, so when it grows in the back odd direction (let's say),it stops at odd number 1 to obtain T_k = H; while all terms of all sequences in G_n are not successful, so as n increases infinitely, G_n is endlessly expanding at a great speed. From Theorem 4.4, we know that the sequences in H are not identical to each other, and no two sequences have the same term (odd number).So H can be regarded as a set of odd numbers. Similarly, from Theorem 5.7, G_n can also be regarded as a set of odd numbers. From the above, it is obtained that as a set of odd numbers, H is a definite set of odd numbers, and as n increases infinitely, $\lim_{n\to\infty} G_n$ is an indeterminate set of odd numbers.

Theorem 6.1:
Any odd number is successful.

Proof:
Let the set of all odd numbers be Q. Consider H as a set of odd numbers. By Theorem 4.4,the set of all successful odd numbers is H. Suppose there is an odd number that is not successful. And let the set of all not successful odd numbers be G. then $G \cap H = \emptyset$, and $G \cup H = Q$.

From Theorem 5.4 and Theorem 5.7, suppose there is an odd number that is not successful, then the sequence set G_n, which is regarded as the set of odd numbers, can be obtained. And it is known that any odd number of G_n is not successful. And from the above remark (3), $\lim_{n\to\infty} G_n$ is an indeterminate set of odd numbers Because $G_n \subset G$, where n = 1,2,3,...,so $\lim_{n\to\infty} G_n \subset G$, and the set G of all odd numbers, those are not successful, is also an indeterminate set of odd numbers. And the sets of odd numbers H and Q are both definite sets of odd numbers. So the equation $G \cup H = Q$ does not hold, contradiction. Thus, $G = \emptyset$, H = Q, and any odd number is successful. QED

Remark:
Treat each odd number in the set H as a vertex. Then connect an edge between any two odd numbers (vertices) in H that have a front and back odd number relationship. In particular, connect an edge between odd number 1 and any other odd number in the odd number set H1 except for odd number 1.At this point, H can be regarded as a tree with an odd root 1.We call it the H-tree. This H-tree contains all odd numbers. Because any triple odd number has no front odd number, each odd number in the triple odd number set F is a leaf of this H-tree.

Theorem 6.2:
Any positive integer is successful, i.e., Collatz conjecture holds.

Proof:
See Remark at the top part of
Preliminaries . QED
Problem:
For any odd number p, how to get a integer k with a formula, so that p is k-steps successful, or, $p \in H_k$ in H

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