OPINION

A proof of Fermat's last theorem by relating to monic polynomial properties

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ABSTRACT

Fermat's Last Theorem (FLT) states that there is no natural number set {a, b, c, n} which satisfies $a^n+b^n=c^n$ or $a^n=c^n-b^n$ when $n \geq 3$. In this thesis, we related LHS and RHS of $a^n=c^n-b^n$ to the constant terms of two monic polymials x^n-a^n and $x^n-(c^n-b^n)$. By doing so, we could inspect whether these two polynomials can be identical when $n \geq 3$, *i.e.*, $x^n-a^n=x^n-(c^n-b^n)$, which satisfies $a^n=c^n-b^n$. By inspecting the properties of two polynomials such as factoring, root structures and graphs, we found that x^n-a^n and $x^n-(c^n-b^n)$ can't be identical when $n \geq 3$, except when trivial cases.

Keywords: Polynomials; Fermat's last theorem; Natural number; LHS; RHS

INTRODUCTION

FLT was inferred in 1637 by Pierre de Fermat, and was proved by Andrew John Wiles in 1995 [1]. But the proof is not easy even for mathematicians, requiring more simple proof.

Let *a, b, c, n* be natural numbers, otherwise specified. We related FLT to the following two monic polynomials.

$f(x)=x^n-a^n$ (1.1)

$g(x)=x^n-(c^n-b^n)$ (1.2)

If $f(x)=g(x)$ is possible for $n \geq 3$, $a^n=c^n-b^n$ is satisfied, and FLT is false. But the factoring, root structure and graph properties of $f(x)$ and $g(x)$ do not allow $f(x)=g(x)$ when $n \geq 3$. So, an=cn-bn can't be satisfied for $n \geq 3$.

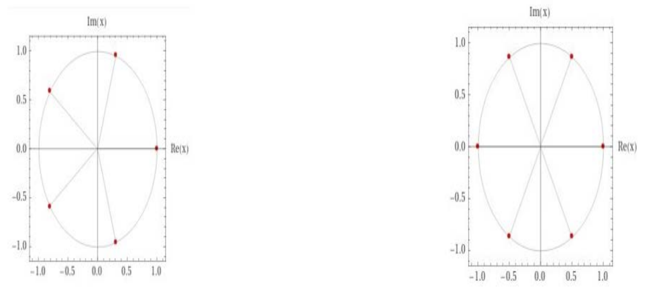
DESCRIPTION

Basic Lemmas

The number of roots of x^n-a^n is as follows, as in Figure 1 [2-4].

1) **Odd $n \geq 3$:** One integer root and n-1 pairwise complex conjugate roots.

2) **Even $n \geq 4$:** Two integer roots and n-2 pairwise complex conjugate roots.



(a) Roots of $x^5-1^n=0$. (b) Roots of $x^6-1^n=0$.

Figure 1) *Number of roots examples of x^n-1^n*

Lemma 2.1. Below (2.1) is the irreducible factoring of (1.1) over the complex field [5].

$f(x)=x^n-a^n=\prod^n_{(k=1)} (x-ae^{(2k\pi i/n)})$ (2.1)

Proof. The n roots of (1.1) are $ae^{(2k\pi i/n)}$, $1 \leq k \leq n$, so, (2.1) is the irreducible factoring of (1.1) over the complex field.

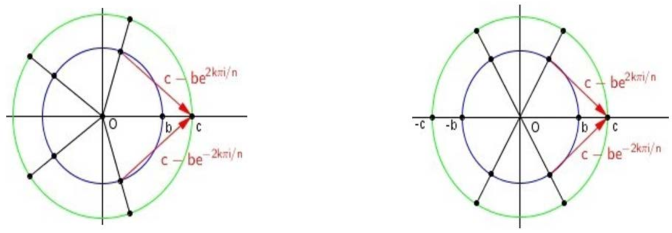
Lemma 2.2. Below (2.2) is the irreducible factoring of $h(c, b)=c^n-b^n$ over the complex field.

$h(c,b)=c^n-b^n=\prod^n_{(k=1)} (c-be^{(2k\pi i/n)})$ (2.2)

Proof. The n roots of $h(c, b)$ are $c=be^{(2k\pi i/n)}$, $1 \leq k \leq n$, so, (2.2) is the irreducible factoring of $h(c, b)$ over the complex field.

Lemma 2.3. All n factors of (2.2) can't have same magnitude.

Proof. The n factors of (2.2) are $c-be^{(2k\pi i/n)}$, $1 \leq k \leq n$. Each factor can be considered as the difference vector between (c, 0) and b(*cos* $2k\pi/n$, *sin* $2k\pi/n$), as in Figure 2.



(a) $n = 5$ example. (b) $n = 6$ example.

Figure 2) *Vector factor examples of (2.2)*

Because $|c - be^{(2k\pi i/n)}|$ is same only with its complex conjugate $|c - be^{(-2k\pi i)/n)}|$, the magnitude of all factors of (2.2) can't be same for all *k*.

Lemma 2.4. A polynomial whose roots are all factors in (2.2) is (2.3) below.

$p(x)=\prod^n_{(k=1)} \{x-(c-be^{(2k\pi i/n)})\}$ (2.3)

Proof. The n factors of (2.2) are $c - be^{(2k\pi i/n)}$, $1 \leq k \leq n$, and they are all involved in (2.3) as individual root. So, $p(x)$ is a polynomial whose roots comprise all factors in (2.2).

Lemma 2.5. A polynomial with different root magnitude can't be of the form x^n-a^n, $n \geq 3$.

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Proof. The n roots of x^n-a^n are all located on a circle of radius a in the complex plane. But, if the magnitude of n roots is not all same, all roots can't be located on a same circle. So, a polynomial with different root magnitude can't be of the form x^n-a^n, $n \geq 3$.

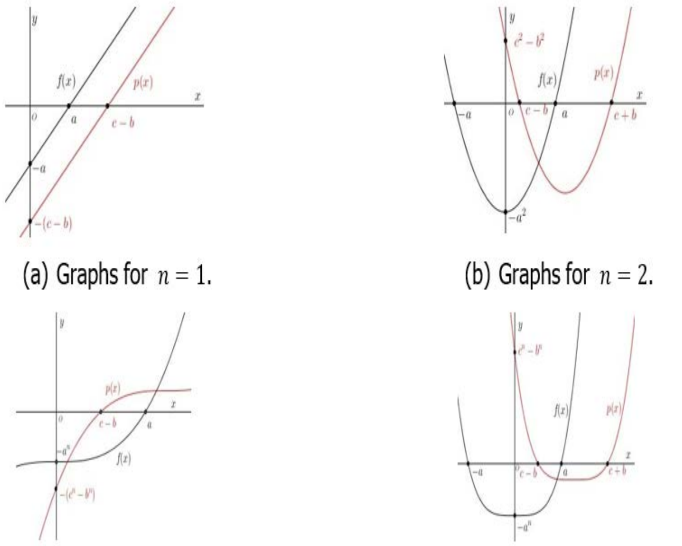
Lemma 2.5 implies that $f(x)$=$g(x)$ can't be achieved for $n \geq 3$, so, a^n=c^n-b^n can't also be satisfied.

Graphical interpretation of FLT and proving lemma

For graphical interpretation of FLT, example graphs of $f(x)$ and $p(x)$ are shown in Figure 3.

$f(x)$=x^n-a^n (1.1)

$p(x)=\prod^n_{(k=1)} \{x-(c-be(2k\pi i/n))\}$ (2.3)



(a) Graphs for $n = 1$.

(b) Graphs for $n = 2$.

(c) Graphs for odd $n \geq 3$.

(d) Graphs for even $n \geq 4$.

Figure 3) *Example graphs of f(x) and p(x)*

We get $f(x)$ by vertically moving y=x^n by -a^n. We get $p(x)$ by horizontally moving y=x^n by c and vertically moving by $-(-b)^n$.

$p(x)=\prod^n_{(k=1)} \{(x-c)-(-be^{2k\pi i/n})\}=\prod^n_{(k=1)} \{X-(-be^{2k\pi i/n})\}=X^n-(-b)^n, X=x-c$ (3.1)

In graph view, FLT is equivalent to the moving of $p(x)$ to overlap $f(x)$, to find possible solutions that satisfy a^n=c^n-b^n. Moving $p(x)$ is equivalent to varying the integer values (b, c), $b \leq a<c$, *i.e.*, moving $p(x)$ vertically or horizontally by integer steps. When any of (b, c) makes two graphs overlap, a solution a^n=c^n-b^n is found, and FLT is false. To make two graphs overlap, the following two steps are required.

1) Horizontal movement that makes X=x-c in (3.1) to be X=x, *i.e.*, c=0.

2) Vertical movement that makes constant terms a^n and c^n-b^n equal.

In Figure 3 (a), when n=1, $p(x)$ always overlaps $f(x)$ for a=c-b. In Figure 3 (b), when n=2, $p(x)$ overlaps $f(x)$ for Pythagorean triples, a^2=c^2-b^2=$(c$-$b)$ $(c+b)$. When n=1, 2, all roots of $f(x)$ and $p(x)$ affect the (x, y) intercepts of the graphs, and there are infinitely many solutions.

But, when $n \geq 3$, the advent of complex roots, which do not appear in graphs, makes situations quite different from those of when n=1, 2. Figure 3 (c) and (d) show that when $p(x)$ overlaps $f(x)$, a=c-b or a^2=c^2-b^2 should be satisfied, which contradicts to a^n=c^n-b^n, $n \geq 3$. This is because the complex roots can't affect the (x, y) intercepts of the graphs. So, any integer step movements of $p(x)$ can't satisfy $p(x)$=$f(x)$ when $n \geq 3$.

When $n \geq 3$, moving $p(x)$ to overlap $f(x)$ is equivalent to making all n roots in $\prod^n_{(k=1)} (c\text{-}be^{2k\pi i/n})$ same as those in $\prod^n_{(k=1)} (ae^{2k\pi i/n})$. Hence Lemma 3.1.

Lemma 3.1. When $n \geq 3$, to make every n roots in $\prod^n_{(k=1)} (c\text{-}be^{2k\pi i/n})$ exactly match to those

$\prod^n_{(k=1)} (ae^{2k\pi i/n})$, c=0, a=-b must be satisfied.

Proof. The complex number identity states that if $x+iy$=$u+iv$, then x=u, y=v [6]. To satisfy $\prod^n_{(k=1)} (ae^{2k\pi i/n})=\prod^n_{(k=1)} (c\text{-}be^{2k\pi i/n})$, keeping all n roots in LHS and RHS identical, $ae^{2k\pi i/n}$=c-$be^{2k\pi i/n}$ must be satisfied.

$a(\cos 2k\pi/n+i\sin 2k\pi/n)$-$c$-$b(\cos 2k\pi/n+i\sin 2k\pi/n)$.

$a\sin 2k\pi/n$-$b\sin 2k\pi/n$, a=-b.

$a\cos 2k\pi/n$-c-$b\cos 2k\pi/n$, c=0.

So, c=0, a=-b.

Lemma 3.1 comprises above mentioned step (1) and step (2), where step (1) makes c=0 and step (2) makes a^n=c^n-b^n=-b^n. That is to say, only trivial solutions can satisfy a^n=c^n-b^n for $n \geq 3$.

CONCLUSION

In this thesis, we related LHS and RHS of a^n=c^n-b^n to the constant terms of two monic polynomials x^n-a^n and x^n-$(c^n$-$b^n)$. By doing so, the proof of FLT is simplified to the proof of whether the two polynomials can be identical when $n \geq 3$. The properties of the two poynomials such as factoring, root structures and graphs showed that x^n-$(c^n$-$b^n)$=x^n-a^n can't be achieved for $n \geq 3$, hence $a^n \neq c^n$-b^n for $n \geq 3$. When n=1, 2, there can be infinitely many x^n-a^n=x^n-$(c^n$- $b^n)$ solutions, but when $n \geq 3$, the advent of the complex roots latches further solutions, except for trivial ones. That is to say, as for the solutions of a^n+b^n=c^n, a+b=c is the first and last solution for odd n, and a^2+b^2=c^2 is the first and last solution for even n.

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