Analytical solution of General Fisher’s Equation by using Laplace Adomian decomposition method

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ABSTRACT

In this paper, we use Laplace Adomian Decomposition Method (LADM) to solve a nonlinear Fisher’s partial differential equation (PDE). The adopted method is illustrated by solving two special cases of Fisher’s partial differential equation. This method is much reliable for its high convergence rate of approximate solutions to the exact solution.

Key Words: One dimension Fisher’s equations, PDEs, Laplace-Adomian decomposition method, Analytical solution

Nonlinear differential equations are too much important to us due to the fact that most of the physical phenomena are nonlinear in nature and which are described by these equations. Particularly partial differential equations play an important role in this regard. In the last few decades several researchers have been made considerable efforts and adopted various approaches for the solution of nonlinear PDEs. Recently in [1], Arqub applied the reproducing kernel Hilbert space method and computational iterative method, in niding the approximate solutions for certain classes of Neumann time-fractional PDEs. In [2], the author studied the approximate solutions for certain classes of Robin time-fractional PDEs via fitting the reproducing kernel algorithm. Similarly we refer to the work in [3,4]. Unfortunately most of the nonlinear problems cannot be solved by analytic methods. Moreover the traditional numerical methods need perturbation, discretization, linearization or transformation to solve the nonlinear problems. The Adomian decomposition method proved free of such steps and hence widely used in the literature, see [5].

Another important method which has gained much concern is the Laplace Adomian polynomial or decomposition method (LADM). It was introduced by Suheil A Khuri [6]. This method is actually a hybrid technique developed by the combination of the two well-known and powerful methods namely Laplace transform and Adomian decomposition method. Using this method one can and the numerical solutions without the restrictive assumptions and discretization and hence it is free from round o errors. Similarly using this method, a solution in the form of in finite series is obtained which has a highest and rapid convergence rate to the exact solution of the concerned problem. By adopting this method, the numerical computations can be reduced. The reliability of LADM and the reductions in computations show that LADM is widely applicable. In addition LADM involving simple and straightforward calculation. This method has applications to the Duffing, Bratu and other nonlinear equations as in [7,8].


Laplace adoman decomposition method and its application to general fisher’s equation

Consider the following general fisher equation

\[
\frac{\partial^2 u}{\partial t^2} - \alpha \frac{\partial u}{\partial t} + \lambda u(a-u)(u-\alpha) = 0.
\]

(1)

We take the operators \(Lu(x,t) = Ru(x,t) + Nu(x,t)\).

From which we represent Eq.(1) as

\[
Lu(x,t) = Ru(x,t) + Nu(x,t).
\]

Taking Laplace transform, we get

\[
\ell[u(x,t)] - u(x,0) = \ell[Ru(x,t)] + \ell[Nu(x,t)]
\]

(2)

\[
\ell[u(x,t)] = \frac{1}{s} f(x) + \frac{1}{s} \ell[Ru(x,t)] + \frac{1}{s} \ell[Nu(x,t)].
\]

Let

\[
u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), Nu = \sum_{k=0}^{\infty} A_k.
\]

Where \(A_k\) is Adomian polynomial which is defined as

\[
A_k = \frac{1}{k!} \frac{d^k}{d\alpha^k} \left[ N \sum_{j=0}^{\infty} A^j u^j \right]_{\alpha=0}.
\]

Then from Eq. (2), we have

\[
\ell\left[ \sum_{k=0}^{\infty} u_k(x,t) \right] = \frac{1}{s} f(x) + \frac{1}{s} \ell\left[ \sum_{k=0}^{\infty} A_k u^j \right] + \frac{1}{s} \ell[\sum_{k=0}^{\infty} A_k].
\]

(3)

Comparing terms on both sides, we have

\[
\ell[U_0(x,t)] = \frac{1}{s} f(x)
\]

\[
\ell[U_1(x,t)] = -\ell[R(U_0(x,t))] + \frac{1}{s} \ell[A_0]
\]

\[
\ell[U_2(x,t)] = -\ell[R(U_1(x,t))] + \frac{1}{s} \ell[A_1]
\]

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Applying inverse Laplace transform, we get

\[ u_0(x,t) = e^{-\frac{t}{s}} \]
\[ u_1(x,t) = e^{-\frac{t}{s}} \]
\[ u_2(x,t) = e^{-\frac{t}{s}} \]
\[ u_{k+1}(x,t) = e^{-\frac{t}{s}} \]

In this way computing inverse transform we get the solution in the form of infinite series as

\[ u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots \]  

Theorem 2.1. Convergence of the method: Suppose that \( E \) is Banach space and \( T: E \to E \) is a contraction nonlinear mapping and if we consider the generated sequence of solutions via LADM is written as

\[ u^{(k)}(x,t) = T(u^{(k-1)}), k = 1, 2, \ldots \]

then the following holds

1. \( \| u^{(k)} - u \| \leq \mu^k \| T u - u \|, 0 \leq \mu < 1; \)
2. \( u^{(k)}(x,t) \) is always hold in the neighborhood \( u(x,t) \) implies that \( u^{(k)} \in B(u,r) \subset E, B(u,r) = \left\{ u^* \in B : \| u^* - u \| < r \right\} \);
3. \( \lim_{k \to \infty} u^{(k)}(t) = u \).

Proof. For proof see [12].

Examples

To demonstrate the applicability of LADM, we consider the following two particular examples of Fisher equations.

Example 3.1 Consider the Fisher equation (17).

\[ \frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} + 1(1-u)(u), \]
\[ u(x,0) = \mu. \]  

Comparing this problem with the general form of Fisher equation in (1), we see that \( \alpha = 1, \theta = 0, \mu = 0 \) and \( f(x) = \mu \). Applying Laplace transform to Eq. (7), we get

\[ \ell [u(x,t)] = \frac{1}{s} \ell \left[ \frac{\partial^2 u}{\partial x^2} \right] + \frac{1}{s} \ell (1-u)(u) \]

\[ = \frac{1}{s} \ell (u(x,t)) \]

By using initial condition, Eq. (8) becomes

\[ u(x,t) = \ell^{-1} \left[ \frac{1}{s} \ell (u(x,t)) \right] + 1 \]
\[ = \ell^{-1} \left( \frac{1}{s} \ell (u(x,t)) + 1 \right) \]

Applying inverse Laplace transform, we get

\[ u(x,t) = \ell^{-1} \left( \frac{1}{s} \ell (u(x,t)) + 1 \right) \]

\[ = u(x,t) = \ell^{-1} \left( \frac{1}{s} \ell (u(x,t)) + 1 \right) \]

The Laplace decomposition method assumes a series solution of the function \( u(x,t) \) is given by

\[ u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), \]

Using Eq. (10) in Eq. (9), we obtain

\[ \sum_{k=0}^{\infty} u_k(x,t) = \ell^{-1} \left( \frac{1}{s} \ell (u(x,t)) + 1 \right) \]

In above equation \( A_k(u) \) is Adomian polynomial which represents the nonlinear term. So Adomian polynomials are given below.

\[ A_0(u) = u^2_0 \]

First few polynomials of the Adomian method are given as:

\[ A_0(u) = u^2_0 \]
\[ A_1(u) = \sum_{k=1}^{\infty} u_k u_{k-1} \]
\[ A_2(u) = \sum_{k=2}^{\infty} u_k u_{k-2} \]

From (11), we get

\[ u_0(x,t) = \mu \]

and the recursive relation:

\[ u_k(x,t) = \ell^{-1} \left( \frac{1}{s} \ell \left( \sum_{j=0}^{k-1} u_j(x,t) \right) + \frac{1}{s} \ell \left( \sum_{j=0}^{k-1} u_j(x,t) \right) + 1 \right) \]

\[ = \ell^{-1} \left( \frac{1}{s} \ell (u(x,t)) + 1 \right) \]

For \( k = 0 \), we get

\[ u_0(x,t) = \mu \]

Therefore, the solution obtained by Laplace Adomian Method is given below

\[ u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) \]
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\[ \frac{\partial u}{\partial t} = \mu \left( 1 + \mu \right) t + \mu \left( 1 - \mu \right) \frac{t^2}{2} + \ldots \]  
(12)

Using algebraic manipulation, we get the exact solution given by

\[ u(x,0) = e^x. \]

Example 3.2.

\[ \left\{ \begin{array}{l}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6(1-u)u,
\end{array} \right. \]

Here \( \alpha = 6; \ u_0 = 0 \) and \( \theta = 0 \) and the initial condition

\[ u(x,0) = e^x. \]

Taking Laplace transform, we have (figure 1)

\[ \ell \left[ u(x,t) \right] = \frac{1}{s}e^x + \frac{1}{s} \ell \left[ \frac{\partial^2}{\partial x^2} u(x,t) \right] + \frac{1}{s} \ell \left[ 6(u-u^2) \right]. \]

Applying the inverse Laplace transform, we get

\[ u(x,t) = e^x + \frac{1}{s} \ell \left[ \sum_{k=0}^{\infty} A_k(u) \right], \]

\( \ell \) \[ \ell \left[ u(x,t) \right] = \ell \left[ e^x \right] + \frac{1}{s} \ell \left[ \frac{\partial^2}{\partial x^2} u(x,t) \right] + \frac{1}{s} \ell \left[ 6(u-u^2) \right]. \]

Applying Eq. (10) in Eq. (13), we obtain

\[ \sum_{k=0}^{\infty} A_k(u) = 6e^x + e^x + \ell \left( \sum_{k=0}^{\infty} A_k(u) \right), \]

\[ + \ell \left[ \frac{1}{s} \ell \left( \sum_{k=0}^{\infty} A_k(u) \right) \right]. \]

Where \( A_k(u) \) are Adomian polynomials few terms of the Adomian polynomials are given below

\[ A_0(u) = 6u_0^2, \]
\[ A_1(u) = 12u_0u_1. \]

From (14), we get

\[ u_0(x,t) = e^x \]

and the recursive relation is

\[ u_{k+1}(x,t) = \ell \left[ \frac{1}{s} \ell \left( \sum_{j=0}^{k} A_j(u) \right) \right] + \frac{1}{s} \ell \left( \sum_{j=0}^{k} u_j(x,t) \right) + \ell \left[ \frac{1}{s} \ell \left( \sum_{j=0}^{k} A_j(u) \right) \right], \]

\( k \geq 0. \)

The few terms of \( u_k(x,t) \) follow

\[ u_1(x,t) = 6e^x + e^x + \ell \left( \frac{1}{s} \ell \left( A_0(u) \right) \right), \]

\[ = 6e^x + e^x + 6 \ell \left( \frac{1}{s} \ell \left( e^x \right) \right). \]

Similarly other terms can be computed. Therefore, the solution obtain by Laplace Adomian decomposition method is given below

\[ u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), \]

\[ = e^x + \left( 7 + 6e^x \right)te^x + \ldots. \]

Which is the required solution (figure 2)

CONCLUSION

Laplace Adomian decomposition method is an hybrid technique consisting of the two different concepts Laplace transform and Adomian decomposition method. This method has some advantages over the previously adopted methods. We can obtain the numerical solutions by this method without the restrictive assumptions and discretization. This method is free from round off errors and it has high convergence rate to the exact solution. In this paper, we have applied the Laplace Adomian decomposition method to general nonlinear Fisher’s partial differential equation; its two particular cases have been successfully handled for their series solutions.

COMPETING INTEREST

It is declared that no competing interest exists regarding this manuscript.
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REFERENCES


