Bifurcation at higher eigenvalues of a class of potential operators and application

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ABSTRACT

We study the existence of a bifurcation branch at the second and higher eigen values of a class of potential operators which possesses the Palais-Smale condition. We will also give an application of our result to a class of semi linear elliptic equations with a critical Sobolev exponent.

Keywords: Hilbert space; Bifurcation point; Potential operator; Higher eigenvalues; Boundary value problem

INTRODUCTION

In E. Tones the author consider a bifurcation at the principal eigenvalue of a class of gradient operators which possesses the Palais-Smale condition. The existence of the bifurcation branch is verified by using the Palais Smale condition [1,2].

Also the author have modified and proved the following basic bifurcation result by Krasnoselskii for gradient mappings by removing the requirement of complete continuity of Φ^J and weak continuity of Φ.

Theorem 1.1: Assume that Φ: $H\rightarrow R$ is weakly continuous and uniformly differentiable in a neighbourhood of 0 and assume that $T=\Phi^J:H\rightarrow H$ is completely continuous. Then, if T is differentiable at 0, every eigenvalue $\mu^*\neq 0$ of the derivative $T^J(0)$ is a bifurcation point for:

T (w)=μw.

We need also the following definition.

Definition 1.2: Let $\Phi \in C^1(H,R)$, and suppose (u_n) is a sequence in H satisfying $\Phi(u_n)\rightarrow c$ and $\Phi j(u_n)\rightarrow 0$ in H^*. Then (u_n) is called a Palais-Smale sequence at level c. If every Palais-Smale sequence at level c contains a strongly convergent subsequence, then Φ is said to satisfy the Palais-Smale condition at level c, and denote by $(PS)_c$.

The main result proved in E. Tones is:

Theorem 1.3: Let Φ^J (u) be a linear operator which is the gradient of a C^1 functional Φ (u). Let Φ^J have a Fre´chet derivative T at the origin in H, where T is a selfadjoint, completely continuous operator [3,4]. Suppose μ_1 is the largest eigenvalue (*i.e.*, $\lambda_1=1/\mu_1$ is the smallest positive characteristic value) of T. Suppose that for some ξ>0 the family of functionals:

$I_\lambda (u)=1/2 \|u\|^2-\lambda\Phi(u)$

Satisfies the $(PS)_c$ condition for λ_1 $\xi<\lambda<\lambda_1+\xi$ and for $c\in R$ in a neighbourhood of 0.

Then λ_1 is a bifurcation point for Φ^J.

We need also the following result

Theorem 1.4: (Spectral theorem for compact operators).

Let T be a compact, selfadjoint linear operator on a infinite dimension separable Hilbert space H. Then H admits an orthonormal basis $(e_n)_n$ consisting of eigenvectors for T [5,6].

The infinite sequence of basis vectors $(e_n)_n$ can be chosen such that the sequence of corresponding eigenvalues (λ_n) decreases numerically,

$|\lambda_1| \geq |\lambda_2| \geq,..., |\lambda_n| \geq ...$ and $\lambda_n\rightarrow\infty$, for $n\rightarrow+\infty$.

Main result

Our main result in this paper is:

Theorem 2.1: Let $\Phi^J(u)$ be a linear operator which is the gradient of a C^1 functional $\Phi(u)$. Let Φ^J have a Frechet derivative T at the origin in H, where T is a positive selfadjoint, completely continuous operator. Suppose μ_2 is the second largest eigenvalue (i.e. $\lambda_2=1/\mu_2$ is the second smallest positive characteristic value) of T. suppose that for some ξ>0 the family of functionals:

$I_\lambda (u) = 1/2 \|u\|^2 - \lambda\Phi(u)$

Satisfies the $(PS)_c$ condition for λ_2 $\xi<\lambda<\lambda_2+\xi$ and for $c \in R$ in a neighbourhood of 0.

Then λ_2 is a bifurcation point for Φ^J.

Proof. Since T is a positive compact, self-adjoint linear operator, then T admits at least one eigenvalue, namely:

$\mu_1=\max\{|(Tu, u)|, u \in H, \|u\|=1\}$·

Choose a corresponding normalized eigenvector $e_1\in H$.

Let $Q_1=\{e_1\}^\perp$ be the orthogonal complement to the 1-dimensional subspace spanned by the vector e_1. Being an orthogonal complement, Q_1 is a closed subspace of the Hilbert space. For $u \in Q_1$, we have the following computation:

$(Tu,e_1)=(u,Te_1)=\mu 1(u,e_1)=0$,

Showing that Q_1 is invariant under T in the sense that $T_u \in Q_1$ if $u \in Q_1$. Hence T can be considered as a compact, self-adjoint linear operator on the Hilbert space Q_1. Working inside Q_1, we then get a second eigenvalue for T,

$\mu_2=\max\{|(Tu,u)|, u \in Q_1, \|u\|=1\}$

and a normalized eigenvector $e_2 \in Q_1$ for μ_2. Clearly, $\mu_1 \geq \mu_2$ and $e_1\perp e_2$.

Now, we consider the operator $T_{|Q1}$: $Q_1\rightarrow Q_1$ and we apply theorem 1.3 for this operator.

We remark that the first eigenvalue of the operator $T_{|Q1}$ correspond to the second eigenvalue of the operator T defined on H.

Remark 2.2: One can generalize the above result to eigenvalues of higher orders.

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Application

We consider the problem presented by E. Tonkes.

Let $\Omega \subset R^N$ a smooth bounded domain and solutions are sought in the Sobolev

Space $H^1{}_0(\Omega)$, endowed with norm $\|u\|=(\int_\Omega |\nabla_u|^2)^{1/2}$,

$-au=\lambda(u+|u|^{2*-2}u)$, on Ω;

$u(x)=0$, $x \in \partial\Omega$.

Solutions to problem (3.1) correspond to critical points of the functional

$J_\lambda(u)=1/2\|u\|^2-\lambda(1/2 \int u^2-1/2^* \int |u|2^*)$

In Tonkes say the functional J_λ satisfies the $(PS)_c$ for some condition on the level c by the same steps as in namely obtained the following lemma.

Lemma 3.1: For $\lambda>0$, J_λ satisfies the $(PS)_c$ condition for $c<c^*{}_\lambda=1/N\ S^{N/2}/\lambda^{(N-2)/2}$.

The following result states that the principal characteristic value of the linear problem $-au=\lambda u)$, $u \in H^1$, is a bifurcation point.

Theorem 3.2: For sufficiently small $r>0$, there exists a solution (λ_r, u_r) to (3.1) with $\|u_r\|=r$. One has $\lambda_r \rightarrow \lambda_0$ as $r \rightarrow 0$.

Consider the linear eigenvalue problem

$-au(x)=\lambda u(x), x \in \Omega$;

$u(x)=0$, $x \in \partial\Omega$. (3.2)

Theorem 3.3: Equation (3.2) has a sequence of eigenvalues λ_n such that

$0<\lambda_1<\lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_n \uparrow +\infty$.

Here we use the convention that multiple eigenvalues are repeated according to their multiplicity. The first eigenvalues λ_1 is simple and the corresponding Eigen functions do not change sign in Ω. Moreover, λ_1 is the only eigenvalue with this property.

We will denote by $\phi 1$ the Eigen function corresponding Eigen function corresponding to λ_1, such that $\phi 1(x)>0$ and $\|\phi 1\|=1$.

From the above result, we can see that each characteristic value λ_n of the Dirichlet problem associated to (3.2) forms a bifurcation point for the problem (3.1). By theorem 2.1 and remark 2.2, one can generalize theorem 3.2 to the following.

Theorem 3.4: For sufficiently small $r>0$, there exists a solution (λ^r, u^r) to (3.1)

with $\|u^r\|=r$. One has $\lambda^r \rightarrow \lambda_n$ as $r \rightarrow 0$.

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