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Coupled systems of boundary value problems for nonlinear fractional differential equations

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ABSTRACT

In this article, we study coupled systems of boundary value problems for fractional order differential equations. We use the idea of the Generalized matric space to develop necessary and sufficient conditions for uniqueness of positive solutions of the system. We also obtain sufficient conditions for existence of at least one solution via nonlinear differentiation of Leray Schauder type. We include an example to illustrate our main results.

Key Words: *Coupled system; Fractional differential equations; Riemann-liouville boundary conditions; Existence; Uniqueness results*

Recently the theory on existence and uniqueness of solutions of fractional differential equations have attracted much attentions and a large number of research articles on the solvability of nonlinear fractional differential equations are available. We refer to (1-4) and the references therein for some of the recent development in the theory. On the other hand, coupled systems of boundary value problems for non linear fractional differential equations are not well studied and only few results can be found dealing with existence and uniqueness of solutions (5-8). Su (5) developed sufficient conditions for existence of solutions to the following coupled systems of two point boundary value problems

$$D^{\alpha}u(t) = f(t, v(t), D^{\mu}v(t)),\ D^{\beta}u(t) = g(t, u(t),\ D^{\nu}u(t)),\ 0 < t < 1,$$
$$u(0) = u(1) = v(0) = v(1) = 0,$$

where $1 < \alpha, \beta \le 2, \mu, \nu$ satisfies $\alpha - \mu$ and $\beta - \nu \ge 1$ and $f, g : [0,1] \times R \times R \to R$ are continuous and D is the standard Rieman-Liouville derivative. Wang et al. (8) obtained sufficient conditions for existence and uniqueness of positive solutions to the following coupled systems of nonlinear three-point boundary values problems

$$D^{\alpha}u(t) = f(t, v(t)),\ D^{\beta}v(t) = g(t, u(t)),\ 0 < t < 1$$
$$u(0) = 0,\ v(0) = 0,\ u(1) = au(\eta), v(1) = bv(\eta),$$

where $1 < \alpha, \beta \le 2, 0 \le a, b \le 1$ and $0 < \eta < 1$ and $f, g : [0,1] \times [0,\infty) \to [0,\infty)$ are continuous.

Motivated by the above studies, we develop some new existence and uniqueness results for the following coupled systems of nonlinear boundary values problems

(1.1)

$$D^{\alpha}u(t) = f(t, u(t), v(t)),\ D^{\beta}v(t) = g(t, u(t), v(t)),\ 0 < t < 1,$$
$$I^{3-\alpha}u(0) = D^{\alpha-2}u(0) = u(1) = 0,\ I^{3-\beta}v(0) = D^{\beta-2}v(0) = v(1) = 0,$$

where $2 < \alpha, \beta \le 3$ and $f, g : I \times R \times R \to R$ are continuous and D, I denote Riemann-Liouville's fractional derivative and fractional integral respectively. We use Perov fixed point theorem (9) and Leray-Schauder fixed point theorem to obtain sufficient conditions for existence and uniqueness results. We also provide an example to illustrated our results.

Preliminaries

We recall some fundamental results and definitions (10,11).

Definition 2.1

The fractional integral of order $\alpha \in R^{+}$ of a function $y : (0,\infty) \to R$ is defined by

$$I^{\alpha}y(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds$$

provided the integral converges.

Definition 2.2

The Riemann-Liouville fractional order derivative of a function $y : (0,\infty) \to R$ is defined by

$$D^{\alpha}y(t) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-s)^{n-\alpha-1}y(s)ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ represents the integer part of α provided that the right side is point wise defined on $(0,\infty)$.

Lemma 2.3

The following result holds for fractional derivative and integral

$I^{\alpha}D^{\alpha}y(t) = y(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + c_3t^{\alpha-3} + ... + c_nt^{\alpha-n}$, for arbitrary $c_i \in R,\ i = 1, 2, ..., n.$

Lemma 2.4

(7) Let X be a Banach space with $\mathcal{P} \subseteq X$ closed and convex. Let Ω be a relatively open subset of $\mathcal{P}$ with $0 \in \Omega$ and $T : \overline{\Omega} \to \Omega$ be a continuous and compact(completely continuous) mapping. Then either

1. The mapping T has a fixed point in $\overline{\Omega}$ or
2. There exist $\mu \in \partial\Omega$ and $k \in (0,1)$ with $\Omega = kTu.$

Definition 2.5

For a nonempty set Z, a mapping $d : Z \times Z \to R^n$ is called a generalized metric on Z if the following hold

$(M1)$ $d(u,v) = 0_{R^n} \Leftrightarrow u = v, \forall u, v \in X;$

$(M2)$ $d(u,v) = d(v,u), \forall u, v \in X,$ (symmetric property);

$(M3)$ $d(x,y) \le d(x,v) + d(v,u) + d(u,v), \forall x, y, u, v, \in X,$ (tetrahedral inequality).

Note: The properties such as convergent sequence, cauchy sequence, open/closed subset are the same for generalized metric spaces as hold for the usual metric spaces.

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Definition 2.6

For an $n \times n$ matrix A, the spectral radius is defined by $\rho(A) = \max\{|\lambda_i|, i = 1, 2, ..., n\}$, where $\lambda_i, (i = 1, 2..., n)$ are the eigenvalues of the matrix A.

Lemma 2.7

(11), Let (Z, d) be a complete generalized metric space and let $T : Z \to Z$ be an operator such that there exist a matrix $A \in M$ with $d(Tu, Tv) \leq Ad(u, v)$, for all $u, v \in Z$. If $\rho(A) < 1$, then T has a fixed point $\mathcal{Z}^* \in Z$, further for any $\mathcal{Z}_0$ the iterative sequence $\mathcal{Z}n + 1 = T\mathcal{Z}n$ converges to $\mathcal{Z}_0$.

Lemma 2.8

An equivalent Fredholm integral representation of the system of boundary value problems (1.1) is given by

(2.1)

$u(t) = \int_0^1 \mathcal{G}_1(t,s) f(s, u(s), v(s)) ds,\ v(t) = \int_0^1 \mathcal{G}_2(t,s) g(s, u(s), v(s)) ds$, where $\mathcal{G}_1, \mathcal{G}_2$ are Green's functions given by

$$(2.2)\quad \mathcal{G}_1(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)} - \dfrac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leq t \leq s \leq 1, \end{cases}$$

$$(2.3)\quad \mathcal{G}_2(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\beta-1}}{\Gamma(\beta)} - \dfrac{(t-s)^{\beta-1}}{\Gamma(\beta)}, & 0 \leq s \leq t \leq 1, \\ \dfrac{[t(1-s)]^{\beta-1}}{\Gamma(\beta)}, & 0 \leq t \leq s \leq 1. \end{cases}$$

Proof

Applying the operator I^α on the first equation of (1.1) and using lemma (2.3), we have

$$(2.4)\quad u(t) = -I^\alpha f(t, u(t), v(t)) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3},\ c_1, c_2, c_3 \in R.$$

The boundary conditions $I^{3-\alpha} u(0) = D^{\alpha-2} u(0) = u(1) = 0$ $c_3 = 0$, $c_2 = 0$ and $c_1 = I^\alpha f(1, u(1), v(1))$.

Hence, (2.4) takes the form

(2.5)

$$u(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, u(s), v(s)) ds - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, u(s), v(s)) ds$$
$$= \int_0^1 \mathcal{G}_1(t,s) f(s, u(s), v(s)) ds.$$

Similarly, by the same process with the second equation of the system, we obtain the second part of (2.1).

Lemma 2.9

(6) The Green's function $\mathcal{G} = (\mathcal{G}_1, \mathcal{G}_2)$ of the system (2.1) has the following properties

(P_1) $\mathcal{G}(t,s)$ is continuous function on the unit square for all $(t,s) \in [0,1] \times [0,1]$;

(P_2) $\mathcal{G}(t,s) \geq 0$ for all $(t,s) \in [0,1]$ and $\mathcal{G}(t,s) > 0$ for all $(t,s) \in (0,1)$;

(P_3) $\max_{0 \leq t \leq 1} \mathcal{G}(t,s) = \mathcal{G}(1,s)$, $\forall s \in [0,1]$;

(P_4) there exist a constant $\gamma \in (0,1)$ such that

$\min_{t \in [\theta, 1-\theta]} \mathcal{G}(t,s) \geq \gamma(s) \mathcal{G}(1,s)$ for $\theta \in (0,1), s \in [0,1]$ where $\gamma = \min\{\gamma_\alpha, \gamma_\beta\}$.

Existence of positive solutions

Define $U = \{u(t) \mid u(t) \in C[0,1]\}$ endowed with the Chebychev norm $\|u\|_\infty = \max_{t \in [0,1]} |u(t)|$. Further, define the norms $\|(u,v)\|_{u \times V} = \|u\|_\infty + \|v\|_\infty$ and $|(u,v)|_{U \times V} = \max\{\|u\|_\infty, \|v\|_\infty\}$. Then, the product spaces $(U \times V, \|.\|)_{U \times V})$, $(U \times V, |.|_{U \times V})$ are Banach spaces. Define the cones $\mathcal{P}\,\mathcal{K} \subset U \times V$ by

$\mathcal{P} = \{(u,v)\} \in U \times V : u(t), v(t) \geq 0, \forall t \in [0,1]\}$ and

$\mathcal{K} = \{(u,v) \in \mathcal{P} : \min_{t \in J}[u(t) + v(t) \geq \gamma \|u, v\|_{U \times V}\}$, where $J = [\theta, 1-\theta]$, $\theta \in (0,1)$.

Lemma 3.1

Assume that $f, g : [0,1] \times R \times R \to R$ are continuous. Then $(u,v) \in U \times V$ is a solution of (2.1), if and only if $(u,v) \in U \times V$ is a solution of system of Fredholm integral equations (1.1).

Proof

The proof of lemma (3.1) is similar to proof of lemma (3.1) in (6).

Define $T : U \times V \to U \times V$ by

(3.1)

$$T(u,v)(t) = \left(\int_0^1 \mathcal{G}_1(t,s) f(s, u(s), v(s)) ds,\ \int_0^1 \mathcal{G}_2(t,s) g(s, u(s), v(s)) ds,\right)$$
$$= (T_1(u,v)(t), T_2(u,v)(t)).$$

By lemma (3.1) the problem of existence of solutions of the integral equations (2.1) coincide with the problem of existence of fixed points of T.

Lemma 3.2

Assume that $f, g : [0,1] \times [0,\infty) \times [0,\infty) \to [0,\infty)$ are continuous. Then $T(\mathcal{P}) \subset \mathcal{P}$ and $T(\mathcal{K}) \subset \mathcal{K}$, where T is defined by (3.1).

Proof

The relation $T(\mathcal{P}) \subset \mathcal{P}$ easily follow from the properties (P_1) and (P_2) of lemma (2.9) and all we need to show that $T(\mathcal{K}) \subset \mathcal{K}$ holds. For $(u,v) \in \mathcal{K}$, we have $T(u,v) \in \mathcal{P}$ and in view of property (P_4) of lemma (2.9), for all $t \in J$, we obtain

(3.2)

$$T_1(u(t), v(t)) = \int_0^1 \mathcal{G}_1(t,s) f(s, u(s), v(s)) ds \geq \gamma_\alpha \int_0^1 \mathcal{G}_1(1,s) f(s, u(s), v(s)) ds.$$

Hence, it follows that

$$\min_{t \in J} T_1(u(t), v(t)) \geq \gamma_\alpha \|T_1(u,v)\|_\infty, \forall t \in J.$$

Similarly, we obtain

$$\min_{t \in J} T_2(u(t), v(t)) \geq \gamma_\beta \|T_2(u,v)\|_\infty, \forall t \in J.$$

It follows that

$$\min_{t \in J}[T_1(u(t), v(t)) + T_2(u(t), v(t))] \geq \gamma \|T_1(u,v), T_2(u,v)\|_{U \times V}, \forall t \in J,$$

which implies that $T(u,v) \in \mathcal{K}$.

Lemma 3.3

Assume that $f, g : [0,1] \times R \times R \to R$ are continuous then $T : \mathcal{P} \to \mathcal{P}$ is completely continuous.

Proof.

We omit the proof, because it is similar to the proof of lemma (3.2) in (6).

Lemma 3.4

Assume that f and g are continuous on $[0,1] \times R \times R \to R$ and there exist $\phi_i \psi_i (i = 1, 2) : (0,1) \to [0, \infty)$ such that the following hold

(H_1) $|f(t,u,v)-f(t,\bar{u},\bar{v})|\le \phi_1(t)|u-\bar{u}|+\psi_1(t)|v-\bar{v}|, t\in(0,1)$, for $u,v,\bar{u},\bar{v}\ge 0$;

(H_2) $|g(t,u,v)-g(t,\bar{u},\bar{v})|\le \phi_2(t)|u-\bar{u}|+\psi_2(t)|v-\bar{v}|$, for $t\in(0,1), u,u,\bar{u},\bar{v}\ge 0$;

(H_2) $\rho(A)<1$, Where the matrix $A\in M_{2,2}(R_+)$ is defined by

$$A=\begin{bmatrix}\int_0^1 \mathcal{G}_\alpha(1,s)\phi_1(s)ds & \int_0^1 \mathcal{G}_\alpha(1,s)\psi_1(s)ds\\ \int_0^1 \mathcal{G}_\beta(1,s)\phi_2(s)ds & \int_0^1 \mathcal{G}_\beta(1,s)\psi_2(s)ds\end{bmatrix}.$$

Then the system (2.1) has a unique positive solution $(u,v)\in\mathcal{P}$.

Proof

Define a generalized metric $d:U\times V\times U\times V\to R^2$ by

$$d((u,v),(\bar{u},\bar{v}))=\begin{pmatrix}\|u-\bar{u}\|_\infty\\ \|v-\bar{v}\|_\infty\end{pmatrix}, \forall((u,v),(\bar{u},\bar{v})\in U\times V.$$

Obviously $(U\times V,d)$ is a generalized complete metric space. For any $(u,v),(\bar{u},\bar{v})\in U\times V$ using the property (P_3) and (H_3), we obtain

$$|T_1(u,v)(t)-T_1(\bar{u},\bar{v})(t)|\le \max_{t\in[0,1)}\int_0^1|\mathcal{G}_1(t,s)|\left[|f(s,u(s),v(s))-f(s,\bar{u}(s),\bar{v}(s))|\right]ds$$

$$\le \int_0^1 \mathcal{G}_1(1,s)\left[\phi_1(s)\|u-\bar{u}\|_\infty+\psi_1(s)\|v-\bar{v}\|_\infty\right]ds$$

$$\Rightarrow\ |T_1(u,v)-T_1(\bar{u},\bar{v})|\le\left(\int_0^1 \mathcal{G}_1(1,s)\phi_1(s)ds\right)\|u-\bar{u}\|_\infty+\left(\int_o^1 \mathcal{G}_1(1,s)\psi_1(s)ds\right)\|v-\bar{v}\|_\infty.$$

Similarly, we obtain

$$|T_2(u,v)-T_2(\bar{u},\bar{v})|\le\left(\int_0^1 \mathcal{G}_2(1,s)\phi_2(s)ds\right)\|u-\bar{u}\|_\infty+\left(\int_o^1 \mathcal{G}_2(1,s)\psi_2(s)ds\right)\|v-\bar{v}\|_\infty.$$

Hence, it follows that

$|T(u,v)=T(\bar{u},\bar{v})|\le Ad((u,v),(\bar{u},\bar{v})),\forall(u,v),(\bar{u},\bar{v})\in U\times V$, Where

$$A=\begin{bmatrix}\int_0^1 \mathcal{G}_\alpha(1,s)\phi_1(s)ds & \int_0^1 \mathcal{G}_\alpha(1,s)\psi_1(s)ds\\ \int_0^1 \mathcal{G}_\beta(1,s)\phi_2(s)ds & \int_0^1 \mathcal{G}_\beta(1,s)\psi_2(s)ds\end{bmatrix}.$$

(H_3), $\rho(A)<1$. Hence by lemma(2.7), the system (2.1) has a unique positive solutions.

Lemma 3.5

Let f and g are continuous on $[0,1]\times R\times R\to R$ and there exist $a_i,b_i,c_i (i=1,2):(0,1)\to[0,\infty)$ satisfying

(H_4) $|f(t,u(t),v(t))|\le a_1(t)+b_1(t)|u(t)|+c_1(t)|v(t)|, t\in(0,1), u,v\ge 0$;

(H_5) $|g(t,u(t),v(t))|\le a_2(t)+b_2(t)(|u(t)|+c_2|v(t)|, t\in(0,1), u,v\ge 0$;

(H_6) $\mathbb{A}_1=\int_0^1 \mathcal{G}_1(1,s)\alpha_1(s)ds<\infty, \mathbb{B}_1=\int_0^1 \mathcal{G}_\alpha(1,s)[b_1(s)+c(s)]ds<1$;

(H_7) $\mathbb{A}_2=\int_0^1 \mathcal{G}_2(1,s)\alpha_2(s)ds<\infty, \mathbb{B}_2=\int_0^1 \mathcal{G}_2[b_2(s)+c_2(s)]ds<1$.

Then the system (2.1) has at least one positive solution (u,v) in

$$\mathcal{Q}=\left\{(u,v)\in\mathcal{P}:(u,v)<\min\left(\frac{\mathbb{A}_1}{1-\mathbb{B}_1},\frac{\mathbb{A}_1}{1-\mathbb{B}_2}\right)\right\}.$$

Proof

Choose $r=\min\left(\frac{\mathbb{A}_1}{1-\mathbb{B}_1},\frac{\mathbb{A}_2}{1-\mathbb{B}_2}\right)$ and define $\Omega=\{(u,v)\in\mathcal{P}:\|(u,v)\|<r\}$.

By lemma (3.3), the Operator $T:\bar{\Omega}\to\mathcal{P}$ is completely continuous. Choose $\mathcal{K}\in$ and $(u,v)\in\partial\Omega$ such that $(u,v)=\mathcal{K}T(u,v)$.

Then, by properties $(P_1),(P_3)$ *and* (H_4), we obtain for all $t\in[0,1]$

$$\|u(t)\|_\infty\le\max_{t\in[0,1]}\int_0^1 \mathcal{G}_1(t,s)|f(s),u(s),v(s)ds|$$

$$\le\mathcal{K}\left[\int_0^1 \mathcal{G}_1(1,s)a_1(s)+\int_0^1 \mathcal{G}_1(1,s)(b_1(s)|u(s)|+c_1(s)|v(s)|)\right]ds\le\mathcal{K}(\mathbb{A}_1+r\mathbb{B}_1)\le\mathcal{K}r.$$

Similarly, we obtain $\|v\|_\infty\le\mathcal{K}r$, hence $\|(u,v)\|_{U\times V}<r$, which shows that $(u,v)\notin\partial\Omega$. Thus by Schauder fixed point theorem, T has a fixed point in $\bar{\Omega}$.

Examples

Example 4.1

Consider the following coupled systems of boundary value problems

$$\begin{cases}D^{\frac{5}{2}}u(t)+\Gamma(\frac{5}{2})\left[\frac{tu(t)}{16}+\frac{t^2v(t)}{32}\right]=0, t\in[0,1],\\ D^{\frac{5}{2}}v(t)+\Gamma(\frac{5}{2})\left[\frac{9t^2|\cos(u(t))|}{16\sqrt{\pi}}+\frac{9t|\cos(v(t))|}{32\sqrt{\pi}}\right]=0, t\in[0,1],\\ I^{\frac{1}{2}}u(0)=D^{\frac{1}{2}}u(0)=u(1)=0,\ and\ I^{\frac{1}{2}}v(0)=D^{\frac{1}{2}}v(0)=v(1)=0.\end{cases}$$

Here

$$\phi_1(t)=\Gamma(\tfrac{5}{2})\tfrac{t}{16},\ \psi_1(t)=\Gamma(\tfrac{5}{2})\tfrac{t^2}{32},\ \phi_2(t)=\Gamma(\tfrac{5}{2})\tfrac{9t^2}{16\sqrt{\pi}},\ \psi_2(t)=\Gamma(\tfrac{5}{2})\tfrac{9t}{32\sqrt{\pi}}.$$

Moreover

$$A=\begin{bmatrix}\int_0^1 \mathcal{G}_1(1,s)\phi_1(s)ds & \int_0^1 \mathcal{G}_1(1,s)\psi_1(s)ds\\ \int_0^1 \mathcal{G}_2(1,s)\phi_2(s)ds & \int_0^1 \mathcal{G}_2(1,s)\psi_2(s)ds\end{bmatrix}=\begin{bmatrix}0.0460 & 0.0007\\ 0.0068 & 0.0058\end{bmatrix}.$$

Here, $\rho(A)=4.61\times10^{-2}<1$, hence by lemma(3.4) the BVP(4.1) has a unique solution. For f and g, we have

$$a_1(t)=0,\ b_1(t)=\Gamma(\tfrac{5}{2})\tfrac{t}{16}, c_1(t)=\Gamma(\tfrac{5}{2})\tfrac{t^2}{32}, a_2(t)=0, b_2(t)=\Gamma(\tfrac{5}{2})\tfrac{9t^2}{16\sqrt{\pi}}, c_2(t)=\Gamma(\tfrac{5}{2})\tfrac{9t}{32\sqrt{\pi}}$$

and by simple calculation, we obtain

$$\mathbb{A}_1=\int_0^1 \mathcal{G}_1(t,s)a_1(s)ds<\infty,\ \mathbb{B}_1=\int_0^1 \mathcal{G}_1(t,s)[b_1(s)+c_1(s)]ds<1,$$

$\mathbb{A}_2=\int_0^1 \mathcal{G}_2(t,s)a_2(s)ds<\infty,\ \mathbb{B}_2=\int_0^1 \mathcal{G}_2(t,s)[b_2(s)+c_2(s)]ds<1$. Hence by using lemma (3.5), BVP (4.1) has at least one positive solution.

CONCLUSION

With the help of Banach theorem and nonlinear Leray Schauder type, we have developed an existence theory to a coupled system of nonlinear FDEs. The concerned results have been successfully obtained and demonstrated by suitable example.

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