Feynman path integral using Lebesgue-Bochner-Stieltjes Integration

Reza R Ahangar[1], Erbil Cetin[2] , Serife Muge Ege[2]

Ahangar RR, Cetin E, Ege SM. Feynman path integral using, Lebesgue-Bochner- Stieltjes integration. J Pure Appl Math. 2024; 8(1):01-08.

ABSTRACT

It was known as Feynman Path Integrals in quantum physics, and a large part of the scientific community still considers them as a heuristic tool that lacks sound mathematical definition. This paper aims to refute this prejudice by providing an extensive and self-contained description of the mathematical theory of Feynman Path Integration, from the earlier attempts to the latest developments, as well as its applications to quantum mechanics.

According to de Brogli, it was realized that light could in fact show behavioral characteristics of both waves and particles. It was demonstrated that electrons show the same dual behavior of matter which was later extended to atoms and molecules. We shall follow the method of integration by the generalized Bochner-Lebesgue-Steilltjes method of integration in Banach space. We demonstrate that the Feynman Path Integral is consistent and can be justified mathematically with the Bogdan integration approach.

Key words: *Feynman path; Mathematical theory; Waves and particles; Sound; Integration*

INTRODUCTION AND HISTORY OF INTEGRATION THEORY IN BANACH SPACE

The traditional differential and integral calculus, invented and developed by Galileo, Leibniz, and Newton has served almost all aspects of sciences, technology, and engineering for centuries [1-4]. Diverse application to natural and social sciences has caused extensive development in the twentieth century.

A more general integration than the Lebesgue was introduced and published by J. Kurzweil 1957 using the Riemann integral. It was studied in-depth by R. Hanstock [5,6]. The stochastic calculus and stochastic models were published by Mc Shane 1974, followed by a unified approach for integration in 1983 [7-9].

The author used Bogdan differential and integral calculus in Banach space and applied it to optimal control of nonlinear operator differential equations in 1986 [10-12].

A general form of this theory called Bochner-Lebesgue Steiltjes measure and integration, will be presented in the ...first chapter. The Generalized Dynamical Systems satisfying operator differential equations cover all delay, functional, and algebraic differential equations.

Several challenges faced mathematics of the twentieth century.

i. A generalized function named after Dirac, called Dirac's Delta function, did not follow the traditional definition of functions. It is extensively used in all aspects of applications in Sciences, Engineering, Technology, and Mathematics. We plan to study nonlinear operator differential equations which include Dirac's delta function based on Banach space differential and integral calculus [13,14].

ii. The development of the Uncertainty Principle in quantum mechanics by Heisenberg imposed a question for mathematics on how we could explain uncertain physical phenomena that described by probabilistic differential equations. Our approach in this article is to use Bochner-Lebesgue-Steiljes integration to justify Feynman path integration [15].

Some foundation of integration in Banach space

The goal here is to review the integration theory in Banach-space-valued functions. To see an easy connection to all varieties of approaches to the integral we begin with the Riemann integral.

Bogdan presented a new development of the theory of Lebesgue and Bochner spaces of summable functions [16]. His development of the integration theory beyond the classical Riemann integral is important for advancements in modern theory of differential equations, theory of generalized functions, theory of operators, probability, optimal control, and most of all in theoretical physics.

Generalized functions, introduced into mathematics by P. Dirac and put on precise mathematical footing by L. Schwartz and Dunford; turned out to be essential in the analysis of ‡flows of matter endowed with mass [17-19].

We shall follow the method of Bogdanowicz with some modifications to construct a generalized Bochner-Lebesgue-Stielltjes integral of the form $\int u(f,d\mu)$ where u is a bilinear operator acting in the product of Banach spaces, f is a Bochner summable function, and μ is a vector valued measure [20, 21].

[1]*Mathematics Department, Texas A & M University-Kingsville, Texas, USA,* [2,3]*EGE University, Mathematics Department, Bornova, Izmir, Turkey*

Correspondence: Reza R Ahangar, Mathematics Department, Texas A & M University-Kingsville, Texas, USA, e-mail: reza.ahangar@tamuk.edu

Received: 2 Jan, 2023, Manuscript No. puljpam-24-6911, Editor Assigned: 3 Jan, 2023, PreQC No. puljpam-24-6911 (PQ), Reviewed: 5 Jan, 2023, QC No. puljpam-24-6911 (Q), Revised: 7 Jan,2023, Manuscript No. puljpam-24-6911 (R), Published: 31 Jan, 2024, DOI:-10.37532/2752-8081.24.8(1).01-08

Step One: Banach Space of Semi-ring of Binary Operations.

Definition 1.2 (Definition of a Group)
Set V with a binary operation "o" is said to be a group if

a. (V, o) is closed,

b. It is associative,

c. There exists an identity element,

d. Every element has an inverse element in V.

The symbol (V, o) is used for a group G with binary operation "o". It is a semi-group when the last condition (d) does not hold. It is called a Monoid if the first two postulates (a) and (b) are true.

A set V may have another binary operation like $(*)$ such that $(V,*)$ will be closed under this operation and it is also associative.

A triple $(V,o,*)$ is a Semi-Ring if

1) (V,o) is a Monoid,

2) $(V,*)$ is also Monoid,

3) $(V,o,*)$ is distributive of "$*$" over the other operation "o"

.

Boolean ring of binary operation on sets
Assume X is an abstract space with no topology defined. Consider V a family of all subsets of X with two operations like $\cup$ and $\cap$. Define the following Boolean operations. We can consider V to be a power set of X, that is, $V=P(X)$. For every element A and B in V define two binary operations $\oplus$ and $\odot$

$$(A \oplus B) = (A \cup B)\backslash(A \cap B)\, and (A \odot B) = A \cap B$$

It is easy to prove that a triple $(V,\oplus,\odot)$ is a ring. The operation $\otimes$ in V can also be described by

$$(A\oplus B) = (A \bigcup B) \backslash (A \bigcap B) = (A \setminus B) \bigcup (B \setminus A)$$

called a symmetric difference.

Step TWO: Semi-ring of partitioned space in Banach space:

Definition 1.3 (Semi-Ring of Subsets)
In general, a semi-ring of subsets of a set X is a family V of subsets of X such that

1) $\phi \in V$, that is V includes the empty set.

2) The set of the family of subsets is closed under ...finite intersection: $A \,\&\, B \in V \Rightarrow A \bigcap B \in V$

3) It is closed under the symmetric differences: For any $A \,\&\, B \in V$ there exists an integer k and mutually disjoint sets: $B_1, B_2, \ldots B_k \in V$ such that $A \setminus B = \bigcup_{j=1}^{k} B_j$.

Conditions (ii) and (iii) imply that V is closed under a symmetric difference.
$A \Delta B = (A \setminus B) \bigcup (B \setminus A)$. The triple $(V, \Delta, \bigcap)$ is called a Semi-ring.

VOLUME AND MEASURABLE SPACES

Step three: Volume and measurable space

A function v from the semi-ring V to a Banach space Z, that is $v: V \to Z$, is called a volume if it satisfies the following conditions: for every countable family of disjoint sets $A_t \in (V, \Delta, \bigcap), (t \in T)$ such that,

$$A = \cup_T A_t \in V \Rightarrow v(A) = \textstyle\sum_T v(A_t) \tag{2.1}$$

The following function μ given by a formula $\mu(A) = \int c_A \, dv$ for $A \in (V, \Delta, \bigcap)$ is well defined. This function will be called the measure and its value on a set A will be called the measure of the set A. The sum is absolutely convergent and $|\mu|(A) = \sup\left\{\sum_{t \in T} |\mu(A_t)|\right\} < \infty$ for any $A \in (V, \Delta, \bigcap)$, where the supremum is taken over all possible decompositions of the set into form (2.1). A volume measure is positive if it takes on only nonnegative values.

Norm in the Positive Volume Space: Let v be a positive volume measure defined in a semi-ring $(V, \Delta, \bigcap)$.

Define a space M of all volumes $\mu: V \to Z$ such that $\mu|(A)| \le c \cdot v(A)$ for some constant c and all $A \in V$. The least constant number c satisfying this inequality is denoted by $\|\mu\|$.

$$\|\mu\| = \min\left\{c \in R : |\mu(A)| \le c \cdot v(A),\, all\, A \in V\right\} \tag{2.2}$$

Claim: The space $(M, \ ||\mu||)$ is a Banach space.

STEP FOUR: Space of Simple Functions and Bochner SUM
Bogdan SUM and the space of Simple (Basic) Functions: Space of S(Y) denotes the set of all functions of the form,

$$h = y_1 \cdot c_{A_1} + y_2 \cdot c_{A_2} + \ldots + y_k \cdot c_{A_k} \tag{2.3}$$

for all $y_j \in Y$ and $A_i \in V\,(for\, j = 1..k)$. Consider a vector $\vec{y} = < y_1, y_2, y_3, \ldots y_k >$ and demonstrate the vector form of the characteristic function

$$\vec{c} = < c_{A_1}, c_{A_2}, \ldots c_{A_k} >, \qquad where\;\; c_{A_i} = \begin{cases} 1 if t \in A_i \\ 0 if t \in A_i \end{cases}$$

then define,

$$h = < \vec{y}, \vec{c} > = \sum_{i=1}^{i=k} y_1 \cdot c_{A_j}$$

Space of simple functions: In other words

$$S(Y) = \{h \in Y : h = \sum_{i=1}^{k} y_j \cdot c_{A_j}\}$$

Let us denote a vector valued function $|\vec{y}| = (|y_1|, |y_2|, \ldots, |y_k|)$ and vector form of the volume of the set A by

$$\vec{v}(A) = (v(A_1), v(A_2), \ldots v(A_k))$$

then the norm of the simple function h will be defined by

$$\|h\| = <|\vec{y}|, \vec{v}(A)) = \sum_{i=1}^{k} |y_j| \cdot v(A_j) \tag{2.4}$$

STEP Five: Bochner Integral

Bochner Integral Operator: For a fixed bilinear continuous operator from a product of the Banach spaces Y, Z into a Banach space W,

$u \in U : Y \times Z -> W$, *and* $\mu \in M$ is a finite additive function from the semi-ring $\mu : V -> Z$, dominated by the volume cv for some constant $c > 0$. define the operator,

$$\int u(h, d\mu) = \sum_{j=1}^{k} u(y_j, \mu(A_j)) \tag{2.5a}$$

$$= u(y_1, \mu(A_1)) + u(y_2, \mu(A_2) + ... + u(y_k, \mu(A_k)) \tag{2.5b}$$

Bochner Integral: We will define also:

$$\int hdv = \sum_{i=1}^{k} y_i \cdot v(A_i) = \tag{2.6a}$$

$$= y_1 \cdot v(A_1)) + y_2 \cdot v(A_2) + ... + y_k \cdot v(A_k)) \tag{2.6b}$$

These two operators are well defined, that is they are independent of the choices $h \in S(Y)$ in (2.3), where $\|h\| = \int |h|$.

Basic Sequence of Simple Functions: A sequence of functions $S_n \in S(Y)$ is a basic if there exists a sequence $h_n \in S(Y)$ and a constant $M > 0$ such that,

$$s_n = h_1 + h_2 + ... + h_n, \quad \textit{and} \quad \|h_n\| \le 4^{-n} M \tag{2.7}$$

for all n = 1; 2; ...

STEP Six: Space of summable functions

The space of summable functions: The space of a summable function $L(v, Y)$ is the set of all functions f generated by a volume space (X, V, μ) which,

$$L(v, Y) = \{ f : \exists s_n \in S(Y) - basicsuchthat \lim_{n \to \infty} s_n = fa.e \}$$

Bogdan Integration Method: We will define the norm of the function $f \in L(Y)$:

$$||f|| = lim\,||s_n||,$$

Assuming that u is a bilinear operator acting in the product of Banach spaces, we can define the Bogdan integral or generalized Bochner Lebesgue-Steiltjes (B-L-S) by

$$\int u(f, d\mu) = \lim_{n->\infty} \int u(s_n, d\mu), \tag{2.8}$$

where f is a Bochner summable function, and μ is a vector valued measure.

If the operator represents a multiplication by a scalar (as a linear operator), that is $u(\lambda, y) = \lambda y$ for y in Banach space Y, and μ represents a Lebesgue measure on a sigma ring, then the integral $\int u(f, d\mu)$ reduces to the classical Bochner integral $\int f \cdot d\mu$.

$$\int f dv = \lim_{n->\infty} \int s_n dv \tag{2.9}$$

If Y=R, the set of real numbers, then the integral reduces to the classical Lebesgue integral.

Measurable Spaces: Some of the properties of the Bochner-Lebesgue-Steiltjes integrals can be studied in the references: [20, 21].

QUANTUM MECHANICAL INTEGRATION OF THE WAVE-PARTICLE IN BANACH SPACE

Historical introduction to Feynman path integral

Feynman path integrals are ubiquitous in quantum physics, even if a large part of the scientific community still considers them as a heuristic tool that lacks a sound mathematical definition. This paper aims to refute this prejudice by providing an extensive and self-contained description of the mathematical theory of Feynman path integration, from the earlier attempts to the latest developments, as well as its applications to quantum mechanics. We will present a detailed discussion of the general theory of complex integration on infinite dimensional spaces, providing on one hand a unified view of the various existing approaches to the mathematical construction of Feynman path integrals and on the other hand a connection with the classical theory of stochastic processes. Moreover, new topics containing recent applications to several Dynamical systems may be related to or added to this problem. This paper bridges the gap between the realms of stochastic analysis and the theory of Feynman path integration. It is accessible to both mathematicians and physicists.

Feynman path integral and multiple slits and multiple screen experiment

In modern physics, the double-slit experiment is a demonstration that light and matter can display characteristics of both waves and particles; moreover, it displays the fundamentally probabilistic nature of quantum mechanical phenomena. This double slit experiment was first performed, using light, by Thomas Young in 1801 as a demonstration of the particle- wave behavior of light [22]. At that time, it was thought that light consisted of either waves or particles. With the beginning of modern physics, about a hundred years later, it was realized that light could in fact show behavior characteristic of both waves and particles. In 1927, Davisson and Germer demonstrated that electrons show the same behavior, which was later extended to atoms and molecules [23, 24].

In 1940, R.P. Feynman discovered how to express quantum dynamics in terms of the Lagrangian instead of Hamiltonian [25].

To understand the Feynman path integral, we start to review the introduction to the traveling of a particle -wave through a double, triple, or multiple slits. In this experiment the light wave may be passing through one or several walls with slits from the source labeled S to a destination object called O.

The goal of the following description is to be able to justify the mathematics of multi-slit and multi-screen experiments in classical and quantum sense, particularly the mathematical justification of Feynman Path Integral.

In deterministic classical mechanics there will be a unique path for a particle- wave between two points: But in quantum mechanics, the question is, how does the initial state (x_0, t_0) of a particle evolve with time to a final state (x_f, t_f) ?

How do we determine the time-evolution of $|\psi(t)\rangle$ of some initial state $|\psi(t_0)\rangle$.

Some elementary assumptions: We will explore step by step mathematical justification of the path integral similar to the previous section. The following description may look like some repetition.

Step One: Banach Space of Semi-ring of paths in multiple slits and multiple screen.

Set the Space
Let X be the abstract space of all paths of particle–wave in the multi-slit and multi-screen experiment. Each trajectory P_j represents the j-th path, and the energy of the wave particle can be demonstrated by S_j, j = 1, 2, 3, ... With this action partitioned the abstract space X, in a time interval $\Delta t = t_{k+1} - t_k$.

We can consider V to be a power set of X, and $S_j \in V$, that is, $V = P(X)$.

To establish a partition in the abstract space X, we can follow the definition in (1.4) for every element $A = S_j$ *and* $B = S_k$ in V with two binary operations $\oplus$ *and* $\odot$.

Step Two: Semi-ring of partitioned space in Banach space:
To establish a semi-ring $(V, \Delta, \cap)$ we can follow the definition in (1.5) for every path A = Sj and B = Sk in V and symmetric differences with two binary operations.

Step Three: Volume and Measurable Space in Quantum Mechanical Approach:
In every action we may consider some physical characteristic such as position, energy, or momentum of the path St: As a result we define a function v from the semi-ring V to a Banach space Z, that is $v: V \to Z$, is called a volume if it satisfies the following conditions: for every countable family of disjoint sets $S_t \in (V, \Delta, \cap), (t \in T)$ such that

$$S = \cup_T S_t \in V and v(A) = \sum_T v(S_t), where\, S_t \Leftrightarrow \psi(t)$$

Notice that:

1. the set T can be countably infinite.
2. for every path Sj there is an associated wave -particle $\psi(t)$ where in Dirac's quantum language we can demonstrate a ket $|\psi(t)\rangle$ as a column vector (t represents a transpose)

$$|\psi(t)\rangle = [\psi_1(t), \psi_2(t), \ldots, \psi_n(t)]^t \tag{3.1}$$

We define the volume of this wave-particle by $v(S_t) = \sum_{k=1}^{k=n} c_k |\psi_k(t)\rangle$ when the number of slits is ...finite. We have a general case when t is in a continuum space.

The following measure μ given by a formula $\mu(|\psi(t)\rangle) = \int c_\psi dv$ *for* $|\psi(t)\rangle \in (V, \Delta, \cap)$ is well defined where cs is a characteristic function. This function will be called the measure and its value on a set S will be called the measure of the set S. The sum is absolutely convergent and

$$|\mu|(|\psi(t)\rangle = \sup\left\{\sum_{t \in T} |\mu|(|\psi(t)\rangle\right\} < \infty \text{ for any } S \in (V, \Delta, \cap),$$

where the supremum is taken over all possible decompositions of the set in (3.1). A volume-measure is positive if it takes on only nonnegative values.

Norm in the Positive Volume Space: Let v be a positive volume measure defined in a semi-ring $(V, \Delta, \cap)$.

Define a space M of all volumes $\mu: V \to Z$ such that $|\mu(A)| \le c \cdot v(S)$ for some constant c and all $S \in V$. The least constant number c satisfying this inequality is denoted by $\|\mu\|$.

$$||\mu|| = min\{c \in R: |\mu(S)| \le c \cdot v(S), all S \in V\} \tag{3.2}$$

Claim: The space $(M, \|\mu\|)$ is a Banach space. Notice that we are planning to justify the Feynman path integral in a Banach space with infinite dimensional space with a norm $\| \ \|$.

STEP FOUR: Space of Simple Functions and Bogdan SUM
In this step we need to describe the differential element of the action $S(x(t))$. It is assumed that we can approximate the position of a particle- wave object x(t) and its velocity $x'(t)$. It is also possible to approximate the kinetic energy K_E and potential energy (P_E) of a differential element ΔS_i, $i = 1, 2, 3, \ldots$ :::of the particle-wave traveling in the partitioned set A_j selected from the semi-ring space triple $(V, \Delta, \cap)$, thus

$$\Delta S_i \approx (K_K(A_j) - P_E(A_j))\Delta t \tag{3.3}$$

Where $K_K(A_j)$ is the kinetic energy of the path A_j and $P_E(A_j)$ is the potential energy of the same path.

Action on infinitesimal Path ΔS_i:

For every path $j = 1, 2, \ldots$ let P_j, represent the j-th trajectory segment with the energy of S_j such that

$$S = \lim_{\Delta t \to 0} \sum_{j=1}^{n} \Delta S_i = \int_{P_j} dS_i = \int (K_K(A_j) - P_E(A_j))\, dt \tag{3.4}$$

Total action

For each path the energy will be $e^{i\frac{S}{\hbar}} = \cos\left(\frac{S}{\hbar}\right) + i \sin\left(\frac{S}{\hbar}\right)$

Where $\hbar = \frac{h}{2\pi}$. The total ...finite value of energy for all paths can be described by

$$\psi(\bar{\chi}) = A(t) \sum_j \exp[\frac{i}{\hbar} S[x(t)]] \tag{3.5}$$

for all infinite continuum trajectories. This sum is over all possible trajectories and A(t) is independent of any individual paths, therefore it depends only on time [26-31].

Basic functions and Bochner SUM
Using (2.3) we will establish a Bochner SUM of Basic functions in the complex plane such that for every point z_j, belonging to the trajectory of the wave-particle in the complex plane there exists a wave function ψ_j. To establish Bochner SUM of basic functions which is similar to the Riemann sum, we define a set of column vectors,

$\vec{\psi} = <\psi_1, \psi_2, \psi_3, \ldots \psi_k>$ in the complex plane such that every ψ_j is associated to the set A_j.

Space of S(Y) denotes the set of all functions of the form $<\vec{\psi}, \vec{c}>$, for all $\psi_j \in Y \text{ and } A_j \in V \; (for \; j = 1..k)$, and demonstrates the vector form of the characteristic function,

$$\vec{c} = < c_{A_1}, c_{A_2}, \ldots c_{A_k} >$$

For every element $A_j \in (V, \Delta, \cap)$ there exists a vector ψ_j such that by the standard complex point $\psi_j = |z_j| e^{i\varphi j}, \; j = 1, 2, \ldots, k. \; where \; i = \sqrt{-1}$, the imaginary number, and $|z_j|$ is the module of the complex point with the argument φ_j for each vector. We demonstrate that the vector form of the characteristic function $c = (c_{A_1}, c_{A_2}, \ldots c_{A_k})$ then.

$$\psi = <\vec{\psi}, \vec{c}> = \sum_{j=1}^{j=k} \psi_j \cdot c_{A_j} \tag{3.6}$$

Space of Simple Functions

We will define the set of simple functions for quantum integration by:

$$S(Y) = \{\psi \in Y : \psi = \sum_{J=1}^{K} \psi_j \cdot c_{A_j}\} \tag{3.7-a}$$

where the set $A_j \in (V, \Delta, \cap)$ in which the wave particle takes a position z_j in the complex plane. Let us denote a vector valued function $|\vec{z}| = (|z_1|, |z_2|, \ldots, |z_k|)$ and vector valued of the volume space by

$$\vec{v}(A) = < v(A_1).v(A_2), \ldots, v(A_k) > \tag{3.7-b}$$

For every path A_j in the partitioned space (V; Δ; ∩), we define the norm of the wave-particle by

$$||\psi|| = < |\vec{\psi}|, \vec{v}(A) > = \sum_{j=1}^{k} |\vec{\psi}_j| . v(A_j) \tag{3.8}$$

STEP Five: Bochner Integral

To move to the next stage of the path integral, we need to redefine the volume of the energy and momentum of each piece A_j.

An objective operator u can be defined and used for the application of this integral

Bochner integral operator

For fixed $u \in U, \mu \in M$ define the operator

$$\int u(h, d\mu) = \sum_{j=1}^{k} u(\psi_j, \mu(A_j)) \tag{3.9-a}$$

$$= u(\psi_1, \mu(A_1)) + u(\psi_2, \mu(A_2) + \ldots + u(\psi_k, \mu(A_k)) \tag{3.9-a}$$

We will also define a special case when the volume generator is a scalar multiplication of the volume space. Thus relation (3.9) will be in the following form:

$$\int h dv = \sum_{i=1}^{k} \psi_j . v(A_j) = <\vec{\psi}, \vec{v}> \tag{3.10a}$$

$$\psi_1 . v(A_1)) + \psi_2 . v(A_2) + \ldots + \psi_k . v(A_k)) \tag{3.10b}$$

Define the path-integral by

Assuming that we define the volume measure of the sets $A_j \in (V, \Delta, \cap)$ for the total numbers of K slots (in Young Physical experiment) for the volume of the set

$$z_1 = r_1 e^{i\phi_1}, z_2 = r_2 e^{i\phi_2}, \ldots, r_k e^{i\phi k}$$

Where $v(A_j) = |z_j| e^{i\phi_j}, j = 1, 2, \ldots K$ is called the volume of each partition set. These two operators, well defined in (3.5) and (3.6), are independent of the choices $h \in S(Y)$ in (3.4), where $||h|| = \int |h|$.

STEP Six: Space of Summable Functions

In our final conclusion, we present the LSB integration that can be used in path integral. In current application of Lebesgue -Bochner integration, the operator $u \in U$ may represent mass, energy, or momentum.

1. Basic Sequence of Simple Functions: A sequence of functions $h_n \in S(Y)$ is a basic if there exists a sequence $h_n \in S(Y)$ and a constant M>0 such that

$$s_n = |\psi_1\rangle + |\psi_2\rangle + \ldots + |\psi_n\rangle, \text{ and } \| s_n \| \le 4^{-n} M \tag{3.12}$$

for all n = 1; 2; ...

2. The Space of Summable Functions: The space of a summable function L(Y) is the set of all functions f which

$$L(Y) = \{ f : \exists \psi_n \in S(Y) \text{ - basic such that } \lim_{n\to\infty} \psi_n = f \; a.e \}$$

Definition: Given that $f \in L(Y)$. Define the norm, LSB integral, and Bochner integral $\| f \| = \lim_{n\to\infty} \| \psi_n \|$

$$\int u(f, d\mu) = \lim_{n\to\infty} \int u(\psi_n, d\mu) \tag{3.13}$$

$$\int f dv = \lim_{n\to\infty} \int \psi_n dv$$

Assume that u is a bilinear operator acting on the function f which is the limit of the particle-wave sequence Ψ_n. The set of operators can be selected as an operator for Position, Mass, Energy, or Momentum of the wave-particle. If this action is applied in the product of Banach spaces, we can de.ne the path integral by a generalized Bochner Lebesgue - Steiltjes (B-L-S): where f is a Bochner Summable function, and μ is a vector-valued measure.

If the operator represents a multiplication by a scalar (or it is a linear operator), that is $u(\lambda, \psi) = \lambda_\psi$ for y in Banach space Y, and μ represents a Lebesgue measure on a sigma ring, then the integral $\int u(f, d\mu)$ reduces to the classical Bochner integral $\int f . d\mu$.

$$\int u(f, d\mu) = \begin{Bmatrix} lim_{n\to\infty} \int u(\psi_n, d_\mu), & if \; lim_{n\to\infty} \psi_n = f \\ \int f d\mu, if \; u(\lambda, \psi) & \end{Bmatrix} \tag{3.14}$$

We will demonstrate the application of previous sections in Feynman Path Integral.

FEYNMAN PATH INTEGRAL IN BANACH SPACE

This general definition of the integral can be applied to a variety of operators u in (3.14). It may be helpful to present a few examples.

Operators in quantum mechanics: Let us denote a vector $\vec{v} = |\psi\rangle$ in Banach Space Y: Every quantum state $|\psi\rangle$ represents either position, momentum, or energy.

i) Position Wave can be described by a linear combination.

$$|\psi\rangle = \sum_j a_j \psi_j, \; j = 1, 2, 3, \ldots \tag{4.1}$$

for discrete case. For continuous case the wave particle and is demonstrated by:

$$|\psi\rangle = \int \psi(x)\,|x\rangle dx \tag{4.2}$$

which is the position operator of the quantum wave ket vector $|\psi\rangle$ for continuous case.

The position operator $\hat{X}$ can be demonstrated when it is acting on a wave particle $|\psi\rangle$ by $\hat{X}\,|\psi\rangle$. If it is only with x- coordinate, then with Bra -KET notation will be

$$\hat{X}\,|\psi\rangle = |x\psi\rangle \Rightarrow \hat{x}\psi(x) = x\psi(x)$$

The Bra vector applies on the left on continuum bases:

$$\langle\psi\,|\,\hat{X}\,|\,\psi\rangle = \int \psi^* x\psi\, dx$$

where ψ^* represents the complex conjugate. It is interesting to show that the eigenvalue and eigenvector:

$$x\psi(x) = \lambda\psi(x) \Rightarrow (x-\lambda)\psi(x) = 0$$

This relation implies that the wave -particle will be positioned on the x-coordinate all the time except when x =λ. This conclusion indicates that the behavior of the solution around the value x =λ is following Dirac's Deleta function, that is:

$$\hat{X}\,|\psi\rangle = \delta(x-\lambda) \tag{4.3}$$

ii) Momentum Operator: Let $|p\rangle$ represent the momentum action on a position x such that,

$$|\mathrm{p}\rangle = [p_1, p_2, p_3, \ldots p_n, \ldots]^t$$

then the quantum state of the momentum can be described by

$$|\mathrm{p}\rangle = \sum a_j\,|\,p_j\rangle, \; j = 1, 2, 3, \ldots \tag{4.4}$$

Time independent momentum operator

Let $\hat{P}_x = -i\hbar\dfrac{\partial}{\partial x}$ be the time independent momentum operator based on the x-axis and acting on a quantum waveψ:

$$-i\hbar\frac{\partial}{\partial x}(\psi) = p.\psi$$

This relation demonstrates that *Operator. Eigenfunction = eigenvalue. Function*

Since this is a time independent differential equation, we can solve it by separation of variables.

$$\frac{d\psi}{\psi} = -\frac{1}{i\hbar}p\,dx \Rightarrow \ln\psi = \frac{i}{\hbar}p.x$$

$$\psi(x) = \psi(0).\exp\left(\frac{i}{\hbar}p.x\right) \tag{4.5}$$

As a result of (4.5). The momentum action on the position x can be described by $\langle x\,|\,p\rangle = \psi(0).\exp\left(\frac{i}{\hbar}p.x\right)$.

The relation $\langle p'\,|\,x\rangle = \psi(0).\exp\left(-\frac{i}{\hbar}p.x\right)$ can be obtained by complex conjugate. We can write,

$$\langle p'\,|\,p\rangle = \delta(p'-p) \tag{4.6}$$

This is an approach to describe the position, energy, and momentum of the solution of the Schrodinger Equation.

Dirac's Integration Intuitive Approach

Dirac never gave an explicit characterization of his intuitive approach to integration. But all of the principles of his integration were complete. The conclusion theorem presents a final theorem that Dirac's Integral Space (DIS) is isomorphic to the category of Lebesgue Measure Spaces. [30].

Quantum Mechanical Solution Operators

Hamiltonian Operator: In this section, we would like to show a solution operator of Schrodinger equation. Assume that there exists an operator U acting on an initial $|\psi(0)\rangle$ and transforms to a wave at time $|\psi(t)\rangle$ which represents a quantum state satisfying the Schrodinger equation,

$$i\hbar\frac{\partial\,|\,\psi(t)\rangle}{\partial t} = H\,|\,\psi(t)\rangle \tag{4.9}$$

where H is Hamiltonian. Thus, the solution can be described by: $|\psi(t)\rangle = U(t)|\psi(0)\rangle$.

Let us substitute its derivative in the equation (4.9). That is

$$\frac{\partial|\psi(t)\rangle}{\partial t}=\frac{dU}{dt}|\psi(0)\rangle$$

$$i\hbar\frac{dU(t)}{dt}|\psi(0)\rangle=H|\psi(t)\rangle=U(t)|\psi(0)\rangle$$

Simplify this relation and solve the ODE problem within respect to the independent variable t.

$$i\hbar\frac{dU(t)}{dt}=HU(t)\Rightarrow\frac{dU(t)}{dt}=\frac{i}{\hbar}HU(t)$$

$$U(t)=\exp(-\frac{i}{\hbar}Ht) \quad (4.10)$$

Thus, the solution operator of the equation (4.9) will be:

$$|\psi(t)\rangle=\exp(-\frac{i}{\hbar}Ht)|\psi(0)\rangle \quad (4.11)$$

The path integral interpretation:
We can use the general form of the Lebesgue-Bochner-Stieltjes integral of (2.8) and (3.9) using the solution operators (4.10) and (4.11). As a result, if $f=\lim|\psi_n(t)\rangle$ then,

$$\int u(f,d\mu)=\int d\mu\,|\psi_n(t)\rangle=\int d\mu\exp(-\frac{i}{\hbar}Ht)|\psi_n(0)\rangle \quad (4.12)$$

5. DISCUSSIONS ON THE PATH INTEGRAL PROPAGATOR

A particle-wave at position x(t) and momentum p will have a kinetic and potential energy:

$$KE=\frac{p_x^2}{2m}\text{ and }PE=V(x) \quad (5.1)$$

The total energy can be demonstrated by

$$E=KE+PE=\frac{p_x^2}{2m}+V(x) \quad (5.2)$$

The propagator of a quantum system between two points (x'; t') and (x_0; t_0) can be defined by the probability transition amplitude between the wave function evaluated at those points:

$$U[(x',t'),(x_0,t_0)]=\langle\psi(x',t')|\psi(x_0,t_0)\rangle \quad (5.3)$$

If the Hamiltonian carries no explicit time dependence, we can assume t_0 and denote $t=t'-t_0$. The left hand side of the relation (3.22) can be described by $U(x',t;x_0)$. Feynman proposed that the contribution of the time independent trajectory x(t) to the propagator is

$$\exp[\frac{i}{\hbar}S\left[x(t)\right]] \quad (5.4)$$

That is every possible path contributes with equal amplitude to the propagator with a phase related to the classical action. Thus,

$$U(x',t;x_0)=A(t)\sum\exp[\frac{i}{\hbar}S(t)]] \quad (5.5)$$

This article has been intended as a mathematical justification of Feynman path integral using the integration theory in Banach Space. The theory of integration evolved from Riemann, Steiltjes, and Lebesgue throughout past centuries.

Bochner provided the integration theory in Banach spaces. The Feynman path integral was originally motivated and presented heuristically.

Several characteristics of the path integral guide us to plan for rigorous mathematical work.

1. It should work in infinite dimensional space.
2. It should be consistent with Dirac's Integral System.
3. It should be working with a variety of operator differential equations.
4. The position vector can be selected as a complex variable.
5. The nature of the quantum level computation is required to use the Lebesgue- Stieltjes measurable space.

Much research has justified the integration theory based on Hilbert space or Banach space. In Generalized approach, we presented integration called Lebesgue Bochner- Steiltjes and used it in this paper to demonstrate that Feynman Path integral is mathematically consistent theory. This work is an introductory development of Feynman path integral in Lebesgue- Bochner- Steiltjes integral and it is yet to be used in many other applications in theoretical physics.

REFERENCES

1. Whiteside DT. The Mathematical Papers of Isaac Newton. Cambr Unive Pr; 1967. 400.
2. Newton I. Section I of Book I of the Principia, explaining "the method of first and last ratios", a geometrical form of infinitesimal calculus, as recognized both in Newton's time and in modern times - see citations above by L'Hospital. 1970;41.
3. Newton I. Newton's Waste Book (Part 3) (Normalized Version): The Newton Project. 1666.
4. Leibniz GW. Nova Methodus pro Maximis et Minimis. 1684.
5. Kurzweil J. Henstock-Kurzweil Integration: Its Relation to Topological Vector Spaces. Series in Real Analysis. World Scientific Publishing Company. 2000:7.
6. Kurzweil J. Integration Between the Lebesgue Integral and the Henstock-Kurzweil Integral: Its Relation to Locally Convex Vector Spaces. Series in Real Analysis. World Scientific Publishing Company; 2002;8.
7. Henstock R. Lectures on the Theory of Integration. Series in Real Analysis. Sci Pub Com. 1988;1
8. McShane EJ. Unified Integration. Academic Press. 1983.
9. McShane EJ. Stochastic Calculus and Stochastic Models. Academic Press. 1997.
10. Ahangar RR. Existence of Optimal Controls for Generalized Dynamical Systems Satisfying Nonanticipating

Operator Differential Equations [dissertation]. Cath Univ Amer. 1986.

11. Ahangar R. Nonanticipating Dynamical Model and Optimal Control. App Mathe L. 1989;2(1):15-18.
12. Ahangar RR. Optimal Control Solution to Nonlinear Causal Operator Systems with Target State. FCS. 218-23.
13. Banach S. Theory of Linear Operations. Amsterdam: North Holland. 1987.
14. Banach S. Théorie des opérations linéaires. Monographie Matematyczne 1. Warszawa. 1932.
15. Bogdan VM. An Approach to the Theory of Lebesgue-Bochner Measurable Functions and to the Theory of Measure. Math Annalen. 1966;164:251-69.
16. Bogdanowicz MW. A Generalization of the Lebesgue-Bochner-Stieltjes and New Approach to the Theory of Integration. Proc Nat Aca Sci. 1965;53(3):492-98.
17. Dirac PAM. The Principle of Quantum Mechanics. Second Edition. Oxford University Press. 1935.
18. Schwartz L. Mathematics for Physical Sciences. Addison Wesley; 1966.
19. Dunford N, Schwartz JT. Linear Operators, Part I. Interscience. 1958.
20. Bogdanowicz WM. An approach to the theory of integration and theory of Lebesgue-Bochner measurable functions on locally compact spaces. Mathematische Annalen. 1967;171:219-38.
21. Bogdanowicz MW. A Generalization of the Lebesgue-Bochner-Stieltjes and New Approach to the Theory of Integration. Proc Nat Aca Sci. 1965;53(3):492-98.
22. Young T. The Bakerian lecture. Experiments and calculation relative to physical optics. Philosophical Transactions of the Royal Society of London. 1804;94:1-16.
23. Physicists Smash Record For Wave. Particle Duality.
24. Eibenberger S. Matter-wave interference with particles selected from a molecular library with masses exceeding 10000 amu. Physi Chem Phy. 2013;15(35):14696-700.
25. Feynman RP, Hibbs AR. Quantum Mechanics and Path Integrals. Princeton University Press. 1988.
26. Feynman RP. Space-Time Approach to Non-relativistic Quantum Mechanics. Rev Mod Phy. 1948;20.
27. de Broglie L. The reinterpretation of wave mechanics. Foun Phy. 1970;1(1):5-15.
28. de Broglie L. Waves and quanta. Nature. 1923;112(2815):540.
29. de Broglie LV. On the Theory of Quanta. Foundation of Louis de Broglie. 2004.
30. Susilo F. A Theory of Dirac Integral Space and the Isomorphism Between the Category of Such Spaces and the Category of Lebesgue Measure Spaces [dissertation]. Sch Arts & Sci Cath Univ Amer. 1990.
31. Weinberg S. The Quantum Theory of Fields. Cambr Univ Ass. 2005.