RESEARCH

Gell-Mann matrices (strong force interaction) in geometric algebra $Cl_{3,0}$

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ABSTRACT

In this paper, we will find a way to apply the Gell-Mann transformations made by the λ_i matrices using Geometric Algebra $Cl_{3,0}$. And without the need of adding the time as an ad-hoc dimension, but just considering that:

$\hat{\boldsymbol{t}} = \hat{\boldsymbol{x}}\hat{\boldsymbol{y}}\hat{\boldsymbol{z}}$

The transformations are as follows. Considering the original ψ:

$\psi = \psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$

The new ψ' obtained when applying each of the Gell-Mann matrices λ_i is:

$$\psi' = (\lambda_1 \to \psi) = \psi_0 + \psi_y\hat{x} + \psi_x\hat{y} + \psi_{zx}\hat{y}\hat{z} + \psi_{yz}\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$
$$\psi' = (\lambda_2 \to \psi) = \psi_0 + \psi_{zx}\hat{x} - \psi_{yz}\hat{y} - \psi_y\hat{y}\hat{z} + \psi_x\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$
$$\psi' = (\lambda_3 \to \psi) = \psi_0 + \psi_x\hat{x} - \psi_y\hat{y} + \psi_{yz}\hat{y}\hat{z} - \psi_{zx}\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$
$$\psi' = (\lambda_4 \to \psi) = \psi_0 + \psi_z\hat{x} + \psi_x\hat{z} + \psi_{xy}\hat{y}\hat{z} + \psi_{yz}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$
$$\psi' = (\lambda_5 \to \psi) = \psi_0 + \psi_{xy}\hat{x} - \psi_{yz}\hat{z} - \psi_z\hat{y}\hat{z} + \psi_x\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$
$$\psi' = (\lambda_6 \to \psi) = \psi_0 + \psi_z\hat{y} + \psi_y\hat{z} + \psi_{xy}\hat{z}\hat{x} + \psi_{zx}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$
$$\psi' = (\lambda_7 \to \psi) = \psi_0 + \psi_{xy}\hat{y} - \psi_{zx}\hat{z} - \psi_z\hat{z}\hat{x} + \psi_y\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$
$$\psi' = (\lambda_8 \to \psi) = \psi_0 + \frac{1}{\sqrt{3}}\psi_x\hat{x} + \frac{1}{\sqrt{3}}\psi_y\hat{y} - \frac{2}{\sqrt{3}}\psi_z\hat{z} + \frac{1}{\sqrt{3}}\psi_{yz}\hat{y}\hat{z} + \frac{1}{\sqrt{3}}\psi_{zx}\hat{z}\hat{x} - \frac{2}{\sqrt{3}}\psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$

Considering that Gell-Mann matrices do not consider at all the existence I if ψ_0 and ψ_{xyz} , it is possible that we should consider them zero form the beginning. Anyhow, above relations would correspond with the most general case.

We have also worked in the bra-ket product using geometric algebra. For the Euclidean case we have the equation (where the cross sign means reverse and the asterisk means conjugate, both mean the same in $Cl_{3,0}$:

$$\psi^\dagger\psi = \psi^*\psi =$$
$$= (\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{z}\hat{y} + \psi_{zx}\hat{x}\hat{z} + \psi_{xy}\hat{y}\hat{x} + \psi_{xyz}\hat{z}\hat{y}\hat{x})(\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}) =$$
$$= (\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} - \psi_{yz}\hat{y}\hat{z} - \psi_{zx}\hat{z}\hat{x} - \psi_{xy}\hat{x}\hat{y} - \psi_{xyz}\hat{x}\hat{y}\hat{z})(\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}) = \rho + \vec{j}$$

Being ρ the probability density:

$\rho = {\psi_0}^2 + {\psi_x}^2 + {\psi_y}^2 + {\psi_z}^2 + {\psi_{yz}}^2 + {\psi_{zx}}^2 + {\psi_{xy}}^2 + {\psi_{xyz}}^2$

And $\vec{j}$ the fermionic current:

$\vec{j} = 2(\psi_0\psi_y + \psi_x\psi_{xy} - \psi_z\psi_{yz} + \psi_{zx}\psi_{xyz})\hat{x} + 2(\psi_0\psi_y + \psi_x\psi_{xy} - \psi_z\psi_{yz} + \psi_{zx}\psi_{xyz})\hat{y} + 2(\psi_0\psi_z - \psi_x\psi_{zx} + \psi_y\psi_{yz} + \psi_{xy}\psi_{xyz})\hat{z}$

We have made the same in the case of orthogonal buy not orthonormal metric, leading to:

$\psi^\dagger\psi = \psi^*\psi = \rho + \vec{j}$

But in this case:

$\rho = {\psi_0}^2 + {\psi_x}^2 g_{xx} + {\psi_y}^2 g_{yy} + {\psi_z}^2 g_{zz} + {\psi_{yz}}^2 g_{yy}g_{zz} + {\psi_{zx}}^2 g_{zz}g_{xx} + {\psi_{xy}}^2 g_{xx}g_{yy} + {\psi_{xyz}}^2 g_{xx}g_{yy}g_{zz}$

And:

$\vec{j} = 2(\psi_0\psi_x - \psi_y\psi_{xy}g_{yy} + \psi_z\psi_{zx}g_{zz} + \psi_{yz}\psi_{xyz}g_{yy}g_{zz})\hat{x} + 2(+\psi_0\psi_y + \psi_x\psi_{xy}g_{xx} - \psi_{yz}\psi_z g_{zz} + \psi_{zx}\psi_{xyz}g_{zz}g_{xx})\hat{y} + 2(+\psi_0\psi_z - \psi_x\psi_{zx}g_{xx} + \psi_y\psi_{yz}g_{yy} + \psi_{xy}\psi_{xyz}g_{xx}g_{yy})\hat{z}$

It has been also shown that the g-2 issue of the muon could be related to gravitational (non-Euclidean metric) issues without needing another natural force.

The difference of the values of g-2 of the muon are:

$a_m - a_t = 2{,}79E - 09$

And the effect of the non-Euclidean metric on the surface of Earth is:

$g_{xx} - 1 = 1{,}392262E - 09$

As you can check, they are in the same order, being one approximately 2 times the other. So, gravitational effects could indeed affect the g-2 value of the muon on the surface or Earth as commented.

Key Words: *Geometric algebra; Strong force interaction; Gell-Mann matrices, Bra-ket product; Non-Euclidean metric*

INTRODUCTION

In this paper, we will calculate which are the transformations performed by the Gell-Mann matrices but in the realm of the Geometric Algebra $Cl_{3,0}$. We will also check how the bra-ket product is performed using Geometric Algebra. And in the end, we will check the effects of gravitational fields (non-Euclidean metric) in this product and how it could affect the g-2 issue of the muon.

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1. We live in eight dimensions

There is a discipline in mathematics that is called Geometric Algebra also known as Clifford Algebras [1-3]. One curious thing of this Algebra is that if you consider a certain number of spatial dimensional (a certain number of independent vectors), automatically appear other dimensions (or if you want to call them, new degrees of freedom or other entities other than vectors).

In fact, the total number of degrees of freedom in an n-dimensional (understanding n as the number of special dimensions or independent vectors) in Geometric Algebra is:

$$\boldsymbol{Total\ number\ of\ degrees\ of\ freedom} = \mathbf{2^n} \quad (1)$$

If we consider that our world has three spatial dimensions (in Geometric Algebra it is called Cl $_{3,0}$), we will have:

$$\boldsymbol{Total\ number\ of\ degrees\ of\ freedom} = \mathbf{2^3 = 8} \quad (2)$$

And in fact, we can check that this is true Figure 1:

In three dimensions, we have three independent vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$ and $\hat{\boldsymbol{z}}$:

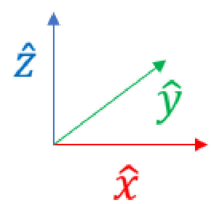

Figure 1) *Basis vectors in three-dimensional space*

In geometric algebra, these three vectors create 5 other entities.

The first other three entities are the bivectors. The bivectors are created multiplying perpendicular vectors. The result of this product is the bivector, an independent entity from the vectors that represent oriented planes. For example, the $\hat{x}\hat{y}$ bivector Figure 2:

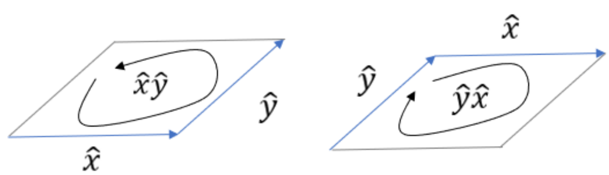

Figure 2) *Representation of the bivectors $\hat{x}\hat{y}$ and $\hat{y}\hat{x}$. They represent the same plane with opposite orientation. In fact, $\hat{x}\hat{y} = -\hat{y}\hat{x}$*

There are three independent bivectors: $\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{y}}\hat{\boldsymbol{z}}$ and $\hat{\boldsymbol{z}}\hat{\boldsymbol{x}}$.

Another appearing entity is the trivector. It is formed by the product of the three independent vectors (and represent an oriented element of volume) Figure 3:

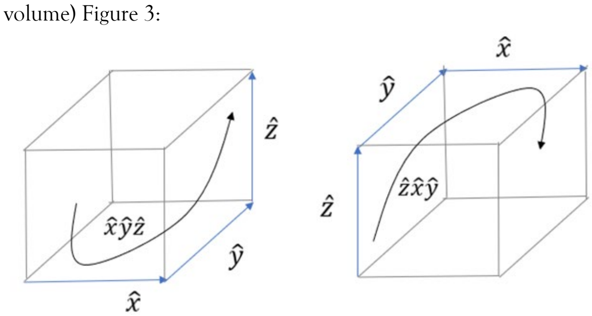

Figure 3) *Representation of the two possible orientations of the trivector*

We can check that $\hat{x}\hat{y}\hat{z} = -\hat{y}\hat{x}\hat{z}$

One important thing of the trivector is that in three dimensions there is only "one trivector". I mean, it can be bigger or smaller or with opposite direction (this means it can be escalated by a real scalar -positive or negative-), but the trivector itself as basis or unit trivector is always the same. You can check Annex A1 to check wheat I am talking about.

Another special property of the bivectors and the trivectors is that the square of a bivector or a trivector is -1. This you can check in all the papers of GA [1-5]. And the square of a vector is 1. Always talking in Euclidean metric. If this is not the case, you can check [2, 4]. That the square of the bivectors and the trivectors is -1, means that they are a clear candidate for the imaginary unit *i* in certain circumstances. And we will see that this property is key for the trivector in the next chapter.

The last entity exiting in Geometric Algebra are the scalars (the numbers). They exist in their own space (are not linear as vectors, surface as bivectors or volume as trivector).

So, in total you can check that we have 8 entities when we have three spatial dimensions: 3 vectors, three bivectors, one trivector and the scalars.

But why are they "degrees of freedom"?

Ok, I will define another concept, the multivector. A multivector is just a sum of all the commented previous entities. This is, for example:

$$\boldsymbol{A} = \boldsymbol{\alpha_0} + \boldsymbol{\alpha_1}\hat{\boldsymbol{x}} + \boldsymbol{\alpha_2}\hat{\boldsymbol{y}} + \boldsymbol{\alpha_3}\hat{\boldsymbol{z}} + \boldsymbol{\alpha_4}\hat{\boldsymbol{x}}\hat{\boldsymbol{y}} + \boldsymbol{\alpha_5}\hat{\boldsymbol{y}}\hat{\boldsymbol{z}} + \boldsymbol{\alpha_6}\hat{\boldsymbol{z}}\hat{\boldsymbol{x}} + \boldsymbol{\alpha_7}\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}\hat{\boldsymbol{z}} \quad (3)$$

Being $\boldsymbol{\alpha_i}$ scalars. This means the multivector (whatever it represents) it has eight degrees of freedom (from $\boldsymbol{\alpha_0}$ to $\boldsymbol{\alpha_7}$). Its meaning can vary a lot depending on the context or the discipline we are talking about.

For example, let us check the position multivector Figure 4:

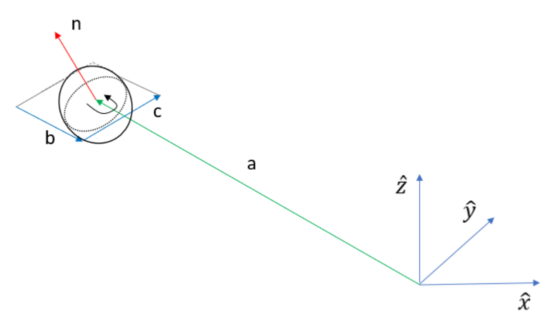

Figure 4) *Representation position multivector*

This multivector has 8 coordinates (8 degrees of freedom corresponding to the scalar, the three space vectors, the three bivectors and the trivector):

$$R = r_0 + r_x\hat{x} + r_y\hat{y} + r_z\hat{z} + r_{xy}\hat{x}\hat{y} + r_{yz}\hat{y}\hat{z} + r_{zx}\hat{z}\hat{x} + r_{xyz}\hat{x}\hat{y}\hat{z} \quad (4)$$

We can see that the vector a in the figure corresponds to the linear position of the particle or to the rigid body center of mass:

$$\vec{a} = r_x\hat{x} + r_y\hat{y} + r_z\hat{z} \quad (5)$$

So, we can simplify the representation of the multivector as:

$$R = r_0 + \vec{a} + r_{xy}\hat{x}\hat{y} + r_{yz}\hat{y}\hat{z} + r_{zx}\hat{z}\hat{x} + r_{xyz}\hat{x}\hat{y}\hat{z} \quad (6)$$

Now let's go to the bivectors. In Figure 4 you can see that there is a bivector $\vec{b}\wedge\vec{c}$ that represents the orientation of a preferred plane in the particle/rigid body. This is, if you select a preferred plane solidary to the particle/rigid body, it tells us the orientation of this plane at a certain time. To define this orientation, you need a coefficient per basis bivector (the same as to define a vector you need the sum of three basis vectors, for bivectors works the same). So:

$$\vec{b}\wedge\vec{c} = r_{xy}\hat{x}\hat{y}+r_{yz}\hat{y}\hat{z}+r_{zx}\hat{z}\hat{x} \quad (7)$$

Introducing in R:

$$R = r_0+\vec{a}+\vec{b}\wedge\vec{c}+r_{xyz}\hat{x}\hat{y}\hat{z} \quad (8)$$

You can see that in a unique multivector R we are having the position and the orientation in the same expression. We have the sufficient degrees of freedom in the expression of the multivector R to give all this information just in one entity (the multivector R).

There are two other components r_0 (the scalar) and $\hat{x}\hat{y}\hat{z}$ (the trivector) that I will explain in the next chapter.

For information, this realm of Geometric Algebra that considers three spatial dimensions, and the eight degrees of freedom (or eight type of elements) created by them is called Geometric Algebra $Cl_{3,0}$.

And for orthonormal bases in Euclidean metric, the following rules apply in $Cl_{3,0}$ [4-6]:

$$\hat{x}^2 = \hat{x}\hat{x} = 1$$
$$\hat{y}^2 = \hat{y}\hat{y} = 1$$
$$\hat{z}^2 = \hat{z}\hat{z} = 1$$
$$\hat{x}\hat{y} = -\hat{y}\hat{x}$$
$$\hat{y}\hat{z} = -\hat{z}\hat{y} \quad (8.1)$$
$$\hat{z}\hat{x} = -\hat{x}\hat{z}$$
$$(\hat{x}\hat{y})^2 = \hat{x}\hat{y}\hat{x}\hat{y} = -\hat{y}\hat{x}\hat{x}\hat{y} = -1$$
$$(\hat{y}\hat{z})^2 = \hat{y}\hat{z}\hat{y}\hat{z} = -\hat{z}\hat{y}\hat{y}\hat{z} = -1$$
$$(\hat{z}\hat{x})^2 = \hat{z}\hat{x}\hat{z}\hat{x} = -\hat{x}\hat{z}\hat{z}\hat{x} = -1$$
$$(\hat{x}\hat{y}\hat{z})^2 = \hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z} = -1$$

It is also important to remark that the scalars and the trivector commute with all the elements. The vectors anticommute among them as you can see in 8.1 equations.

The bivectors anticommute among them. Example:

$$(\hat{x}\hat{y})(\hat{y}\hat{z}) = \hat{x}\hat{y}\hat{y}\hat{z} = \hat{x}\hat{z} = -\hat{z}\hat{x} \quad (8.2)$$
$$(\hat{y}\hat{z})(\hat{x}\hat{y}) = \hat{y}\hat{z}\hat{x}\hat{y} = -\hat{z}\hat{y}\hat{x}\hat{y} = \hat{z}\hat{x}\hat{y}\hat{y} = \hat{z}\hat{x}$$

And the vectors and the bivectors anticommute among them, when the result is a vector. And they commute if the result is $\hat{x}\hat{y}\hat{z}$ (or any of its permutations). Examples:

$$(\hat{x})(\hat{z}\hat{x}) = \hat{x}\hat{z}\hat{x} = -\hat{x}\hat{x}\hat{z} = -\hat{z} \quad (8.3)$$
$$(\hat{z}\hat{x})(\hat{x}) = \hat{z}\hat{x}\hat{x} = \hat{z}$$
$$(\hat{x}\hat{y})(\hat{z}) = \hat{x}\hat{y}\hat{z}$$
$$(\hat{z})(\hat{x}\hat{y}) = \hat{z}\hat{x}\hat{y} = -\hat{x}\hat{z}\hat{y} = \hat{x}\hat{y}\hat{z}$$

It is also worth to comment that the associative and distributive properties apply to all the elements of the multivector (and in general in Geometric Algebra). It is only the commutative and anticommutative properties that apply differently (according above rules).

To sum up:

- The scalars and the vectors have positive square. The bivectors and the trivectors have negative square (see 8.1 equations).
- The scalars and the trivector commute with all the elements. The vectors anticommute above them (equations 8.1). Bivectors anticommute among them (equations 8.2). Vectors and bivectors can anticommute (if the result are a vector) or commute (if the result is the trivector). See equations 8.3.

2. Time as the trivector

I am not going to explain a lot here and the reason is because what you are going to hear is very difficult to believe and digest. You can check papers to check all the info that corroborates what I am going to tell now.

In Geometric Algebra, it is not necessary that the time is a fourth dimension of the space-time (the classical 3 space dimensions and one 4^{th} time dimension).

In Geometric Algebra, the time can be the 8^{th} degree of freedom of the 8 degrees of freedom (or dimensions created by the GA itself). The time is emerging as one of the dimensions that appear automatically when the three spatial dimensions exist.

This is, the basis vector of the time is not a separate vector $\hat{\boldsymbol{t}}$ but it is the trivector $\hat{\boldsymbol{x}}\hat{\boldsymbol{y}}\hat{\boldsymbol{z}}$ already commented.

$$\hat{\boldsymbol{t}} = \hat{\boldsymbol{x}}\hat{\boldsymbol{y}}\hat{\boldsymbol{z}} \quad (9)$$

The main reasons to consider this are:

- The signature of time is negative in General Relativity [7]. This can only be achieved considering an ad-hoc metric with a -1 signature or considering imaginary numbers. In GA, this is not necessary as the basis vector of time (the trivector) has a negative square as expected.
- I have written three papers where it is checked that considering this in Dirac Equation, Maxwell equations and Lorentz Force equations match perfectly (see chapters 4, 5 and 6 of this chapter for more information). In fact, that the spinor of the Dirac equation has 8 degrees of freedom, and to consider one of them, the time-trivector, match perfectly with the equations [8-10].

So, you will check that from this point on, we will consider always the trivector as the basis vector of time. This does not mean that the trivector could not mean other things depending on the context (sometimes, it could be related to spin or to the electromagnetic trivector see chapter 7). The same than a vector can sometimes represent a position, others a force etc. the trivector is just a tool that has certain properties, and these properties match perfectly with the properties of what we perceive as time.

Anyhow, that the trivector represents at the same time the volume and the time could be a hint that somehow, they are related. And the time could be a kind of measurement of the continuous creation of volume in the universe (you can check different mechanisms of creation of volume by the masses in the universe.

After this shock, we continue with the other pending item of the previous chapter, this is, r_0. The meaning of this element is more obscure. As I have commented, the scalars in the multivector are a kind of scalation factor that affects all the magnitudes that are multiplied by it.

So, it could be related to a kind of scalation in the metric appearing in non-Euclidean metrics (kind of local Ricci scalar or trace of the metric tensor).

Another simpler interpretation for r_0, is that the scalars appear when we multiply or divide vectors (or bivectors or the trivector) by themselves. So, sometimes it is necessary a degree of freedom to accommodate these results when they appear. For example, in the current density through time, sometimes is accompanied by the trivector and other times is just a scalar depending on the operations that have been performed before [6].

And to finish, I will just comment the i imaginary unit. In geometric algebra will be always substituted by another that also has -1 square.

These are the bivectors or the trivector (in Cl3,0). When it is related with a magnitude with no preferred direction (energy, time), it will be substituted by the trivector $\hat{x}\hat{y}\hat{z}$. When i is attached to a magnitude with specific direction (velocity, momentum) it will be represented by a bivector (normally representing the plane perpendicular to that direction). You can see examples in [4,5]. In this paper, the substitution by the trivector $\hat{x}\hat{y}\hat{z}$ will be sufficient, as we will see.

3. Spinors in Geometric Algebra $Cl_{3,0}$

If we have a spinor in matrix notation like:

$$\psi = \begin{pmatrix} \psi_{1r} + \psi_{1i} i \\ \psi_{2r} + \psi_{2i} i \\ \psi_{3r} + \psi_{3i} i \\ \psi_{4r} + \psi_{4i} i \end{pmatrix}$$

And we apply the substitution of i by the trivector as commented in chapters 1 and 2.

$$i \to \hat{x}\hat{y}\hat{z}$$

We get:

$$\psi = \begin{pmatrix} \psi_{1r} + \psi_{1i} i \\ \psi_{2r} + \psi_{2i} i \\ \psi_{3r} + \psi_{3i} i \\ \psi_{4r} + \psi_{4i} i \end{pmatrix} = \begin{pmatrix} \psi_{1r} + \psi_{1i}\hat{x}\hat{y}\hat{z} \\ \psi_{2r} + \psi_{2i}\hat{x}\hat{y}\hat{z} \\ \psi_{3r} + \psi_{3i}\hat{x}\hat{y}\hat{z} \\ \psi_{4r} + \psi_{4i}\hat{x}\hat{y}\hat{z} \end{pmatrix} \tag{10}$$

Now, we want to project the spinor in the four dimensions $\hat{x}, \hat{y}, \hat{z}$ and $\hat{t}$ premultiplying by them:

$$(\hat{x} \quad \hat{y} \quad \hat{z} \quad \hat{t})\psi = (\hat{x} \quad \hat{y} \quad \hat{z} \quad \hat{t}) \begin{pmatrix} \psi_{1r} + \psi_{1i}\hat{x}\hat{y}\hat{z} \\ \psi_{2r} + \psi_{2i}\hat{x}\hat{y}\hat{z} \\ \psi_{3r} + \psi_{3i}\hat{x}\hat{y}\hat{z} \\ \psi_{4r} + \psi_{4i}\hat{x}\hat{y}\hat{z} \end{pmatrix}$$

But here we can use the equation (9) of chapter 2:

$$\hat{t} = \hat{x}\hat{y}\hat{z}$$

To get to: $(\hat{x} \quad \hat{y} \quad \hat{z} \quad \hat{t})$

$$\psi = (\hat{x} \quad \hat{y} \quad \hat{z} \quad \hat{x}\hat{y}\hat{z}) \begin{pmatrix} \psi_{1r} + \psi_{1i}\hat{x}\hat{y}\hat{z} \\ \psi_{2r} + \psi_{2i}\hat{x}\hat{y}\hat{z} \\ \psi_{3r} + \psi_{3i}\hat{x}\hat{y}\hat{z} \\ \psi_{4r} + \psi_{4i}\hat{x}\hat{y}\hat{z} \end{pmatrix}$$

Here, you could think that the $\hat{x}\hat{y}\hat{z}$ in the row vector, should be negative as we conjugate (put negative the elements with square -1) when a column vector is converted to row. Even you could think that instead of projecting to $\hat{t}$, we could project to the scalar 1 (as our APS friends probably would propose). No worries, all of them would work, just conventions of signs or nomenclature between elements will change but the result would remain coherent [11-20]. You can make the checking if you want.

If we continue operating:

$$(\hat{x} \quad \hat{y} \quad \hat{z} \quad \hat{t})\psi = (\hat{x} \quad \hat{y} \quad \hat{z} \quad \hat{x}\hat{y}\hat{z}) \begin{pmatrix} \psi_{1r} + \psi_{1i}\hat{x}\hat{y}\hat{z} \\ \psi_{2r} + \psi_{2i}\hat{x}\hat{y}\hat{z} \\ \psi_{3r} + \psi_{3i}\hat{x}\hat{y}\hat{z} \\ \psi_{4r} + \psi_{4i}\hat{x}\hat{y}\hat{z} \end{pmatrix} = \hat{x}(\psi_{1r} + \psi_{1i}\hat{x}\hat{y}\hat{z}) + \hat{y}(\psi_{2r} + \psi_{2i}\hat{x}\hat{y}\hat{z}) + \hat{z}(\psi_{3r} + \psi_{3i}\hat{x}\hat{y}\hat{z}) + \hat{x}\hat{y}\hat{z}(\psi_{4r} + \psi_{4i}\hat{x}\hat{y}\hat{z}) = \psi_{1r}\hat{x} + \psi_{1i}\hat{x}\hat{x}\hat{y}\hat{z} + \psi_{2r}\hat{y} + \psi_{2i}\hat{y}\hat{x}\hat{y}\hat{z}\psi_{3r}\hat{z} + \psi_{3i}\hat{z}\hat{x}\hat{y}\hat{z}$$
$$\psi_{4r}\hat{x}\hat{y}\hat{z} + \psi_{4i}\hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z} = \psi_{1r}\hat{x} + \psi_{1i}\hat{y}\hat{z} + \psi_{2r}\hat{y} + \psi_{2i}\hat{z}\hat{x} + \psi_{3r}\hat{z} + \psi_{3i}\hat{x}\hat{y} + \psi_{4r}\hat{x}\hat{y}\hat{z} - \psi_{4i}$$

If we rename the components the following way:

$$\begin{aligned} \psi_{1r} &= \psi_x \\ \psi_{2r} &= \psi_y \\ \psi_{3r} &= \psi_z \\ \psi_{1i} &= \psi_{yz} \\ \psi_{2i} &= \psi_{zx} \\ \psi_{3i} &= \psi_{xy} \\ \psi_{4r} &= \psi_{xyz} \\ -\psi_{4i} &= \psi_0 \end{aligned} \tag{11}$$

We obtain the typical definition of a spinor in Geometric Algebra $Cl_{3,0}$ [5,20-47].

$$\psi = \psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z} \tag{12}$$

We can see that it has the 8 degrees of freedom commented for a general multivector in Geometric Algebra $Cl_{3,0}$. You can compare it with equations (1) or (1.1) for example.

Although one of the ideas of geometric Algebra is not to use matrices, but as we will have to handle the Gell-Mann matrices (see next chapter) in this paper, we will show also the matrix form of this multivector. So, we have it prepared when we have to use it. We apply equations (1) to (19) to obtain:

$$\psi = \begin{pmatrix} \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \\ \psi_{xyz} - \psi_0\hat{x}\hat{y}\hat{z} \end{pmatrix} \tag{13}$$

In this paper, we will proceed as follows. We will apply the Gell-Mann matrices (see next chapter) to an original spinor called ψ (13). These matrices will transform the original spinor ψ into a new one called ψ' (14) with a similar form but different values for each element.

$$\psi' = \begin{pmatrix} \psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z} \\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z} \\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z} \\ \psi'_{xyz} - \psi'_0\hat{x}\hat{y}\hat{z} \end{pmatrix} \tag{14}$$

Once we have this transformed matrix vector, we can obtain its equivalent multivector in Geometric Algebra $Cl_{3,0}$ as follows:

$$\psi' = \psi'_0 + \psi'_x\hat{x} + \psi'_y\hat{y} + \psi'_z\hat{z} + \psi'_{yz}\hat{y}\hat{z} + \psi'_{zx}\hat{z}\hat{x} + \psi'_{xy}\hat{x}\hat{y} + \psi'_{xyz}\hat{x}\hat{y}\hat{z} \tag{15}$$

4. Gell-Mann matrices

First comment is that it has been already tried to get a correspondence between Gell-Mannatrices and geometric algebra [48-53]. Some of the ideas in the paper have come from these previous papers.

The Gell-Mann matrices used in Strong Force interactions are as following [46]:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$
$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$$
$$\lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$
$$\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

The first thing that we see is that they 3x3 matrices. So how can we use them in matrix vectors of 4 rows as (13):

$$\psi = \begin{pmatrix} \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \\ \psi_{xyz} - \psi_0\hat{x}\hat{y}\hat{z} \end{pmatrix}$$

One of the important things of Gell-Mann matrices is that they only act in anticommutative elements. See ref for more details [46]. In Geometric Algebra $Cl_{3,0}$ (you can check it playing with 8.1,8.2 and 8.3 equations in chapter 2) the anticommutative elements are the vectors $(\hat{x}, \hat{y}, \hat{z})$ and the bivectors $(\hat{y}\hat{z}, \hat{z}\hat{x}, \hat{x}\hat{y})$. And the scalars and trivector $(1, \hat{x}\hat{y}\hat{z})$ are always commutative. See chapter 2 and [1-6] for more details.

If you check equation (12) which is the multivector equivalent to (13)

$$\psi = \psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$

You can see that the coefficients attached to vectors and bivectors (anticommutative) are:

$\psi_x \quad \psi_y \quad \psi_z \quad \psi_{yz} \quad \psi_{zx} \quad \psi_{xy}$

So, this means above coefficients will be affected by the Gell-Mann matrices.

While the commutative ones ψ_{xyz} and ψ_0 will not be affected. Thies leads us to an easy solution. The matrix vector we will use when using Gell-Mann matrices, will be this "cropped" one:

$$\psi = \begin{pmatrix} \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix}$$

Where we have eliminated the last row. And once we make the operation to this vector using Gell-Mann matrices, we just have to add the last row again (or put it as zero? we will comment later this point). What it is clear is that we do not lose any generality not considering some elements that are not affected by a transformation and then putting them again. As commented Gell-Mann matrices only "touch" anticommutative elements, so we do not need the row with commutative elements to perform these operations.

The resultant matrix vector, accordingly, will have this form:

$$\psi' = \begin{pmatrix} \psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z} \\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z} \\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix} \tag{16}$$

And when finish all the operation, we can just add the last row again.

There is another way to manage this situation and it is to use Gell-Mann matrices of 4x4 elements if you prefer that option. This we will comment in chapter 7.

But we will start using the standard Gell-Mann matrices of 3x3 elements. As commented, this means the last row of original spinor:

$$\psi = \begin{pmatrix} \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \\ \psi_{xyz} - \psi_0\hat{x}\hat{y}\hat{z} \end{pmatrix}$$

Will not be touched by the Gell-Mann matrices.

$$\psi' = \begin{pmatrix} \psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z} \\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z} \\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z} \\ \psi'_{xyz} - \psi'_0\hat{x}\hat{y}\hat{z} \end{pmatrix} \tag{17}$$

So, in the transformed spinor, the following equations will always apply (as these elements will not be affected by Gell-Mann matrices:

$$\psi'_{xyz} = \psi_{xyz} \tag{18}$$
$$\psi'_0 = \psi_0$$

There is another possibility as we will comment later in chapters 5 and 7. It is that the lambda matrices in an implicit move, provokes the following:

$$\psi'_{xyz} = 0 \tag{19}$$
$$\psi'_0 = 0$$

This is equivalent as deciding if we add a fourth row and column in Gell-Matrices and decide to put all zeros in these new lines. Or in the case (18) to put all zeros, except a 1 in the diagonal. See chapter 7 for more details.

At this stage we will work with equations (18) for two reasons:

- As the Gell-Mann matrices do not consider those parameters, we guess they are not touched, so they keep the same value.
- And a practical reason. The case (18) is more general. This means, if in the end, we discover that they should be zero, we can just eliminate them from the equations. The opposite case, supposing that they are zero and then going backwards is more complicated.

Anyhow, as will see in the following chapters 5 and 7, the option that they become zero (equation) seem more plausible/symmetric. Anyway, we will stick to equations for the reasons commented above.

5. Applying Gell-Mann matrices to ψ

In this chapter, we will apply each Gell-Mann matrix to ψ to get the result of this transformation as a new ψ'. This is, we will perform the following matrix multiplication:

$$\psi' = \lambda_i\psi = \lambda_i \begin{pmatrix} \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix} = \begin{pmatrix} \psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z} \\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z} \\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix} \tag{20}$$

Where the ψ and the ψ' correspond to the 3-row version of the matrix vector (14) and (16). This way they can be multiplied by the 3x3 Gell-Mann matrices λ_i (See chapter 4). Once this operation is done, we will obtain the relation between the different elements of original ψ and the obtained ψ'.

With this information, we will obtain the new ψ' also in Geometric Algebra $Cl_{3,0}$ representation from the original ψ. This is, we will see which the resultant multivector (15) when λ_i is applied to (12):

$$\psi = \psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$

$$\psi' = \psi'_0 + \psi'_x\hat{x} + \psi'_y\hat{y} + \psi'_z\hat{z} + \psi'_{yz}\hat{y}\hat{z} + \psi'_{zx}\hat{z}\hat{x} + \psi'_{xy}\hat{x}\hat{y} + \psi'_{xyz}\hat{x}\hat{y}\hat{z}$$

5.1 Gell-Mann matrix λ_1

In chapter 4, we saw that λ_1 was:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

If we apply it to equation (20):

$$\psi' = \begin{pmatrix} \psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z} \\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z} \\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix} = \lambda_1\psi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix} = \begin{pmatrix} \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ 0 \end{pmatrix}$$

We obtain:

$\psi'_x = \psi_y$
$\psi'_{yz} = \psi_{zx}$
$\psi'_y = \psi_x$ (21)
$\psi'_{zx} = \psi_{yz}$
$\psi'_z = 0$
$\psi'_{xy} = 0$

And we have to add the two equations (18) commented in the end of chapter 4. As commented, at this stage we will consider them as the ones to apply (with all the considerations commented in chapter 4). Other possibilities as equations (19) or even different could be considered. Apart from what we have commented in chapter 4, we will come back with more comments about this mainly in chapter 7.

$\psi'_{xyz} = \psi_{xyz}$
$\psi'_0 = \psi_0$

Remind that we had defined ψ' (in geometric algebra notation) as:

$$\psi' = \psi'_0 + \psi'_x\hat{x} + \psi'_y\hat{y} + \psi'_z\hat{z} + \psi'_{yz}\hat{y}\hat{z} + \psi'_{zx}\hat{z}\hat{x} + \psi'_{xy}\hat{x}\hat{y} + \psi'_{xyz}\hat{x}\hat{y}\hat{z}$$

This means the new ψ' obtained of the transformation of applying λ_1 lambda1 to ψ is (following all the relations (21) and (18) above):

$$\psi' = \psi_0 + \psi_y\hat{x} + \psi_x\hat{y} + \psi_{zx}\hat{y}\hat{z} + \psi_{yz}\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z} \quad (22)$$

This means, the matrix λ_1 interchanges the coefficients in the $\hat{x}$ and $\hat{y}$ axes and in the $\hat{y}\hat{z}$ and $\hat{z}\hat{x}$ planes. This is, it creates a kind of rotation in these axes/planes of the ψ function. It also destroys all the information regarding $\hat{z}$ axis and $\hat{x}\hat{y}$ planes. And as commented, for ψ_0 and ψ_{xyz} we keep them as not affected unless something different is discovered in the future (other clear possibility would be that they become zero as commented in chapter 4 and will comment later).

Now, that we have seen the process, let's go to the effects in the rest of the Gell-Mann matrices. We will reduce the comments -as everything commented for λ_1 will apply in general. And we will make just comments specific to the different results.

5.2 Gell-Mann matrix λ_2

$$\psi' = \begin{pmatrix}\psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z}\\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z}\\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z}\end{pmatrix} = \lambda_2\psi = \begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z}\\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z}\\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z}\end{pmatrix} =$$

$$\begin{pmatrix}0 & -\hat{x}\hat{y}\hat{z} & 0\\ \hat{x}\hat{y}\hat{z} & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z}\\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z}\\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z}\end{pmatrix} = \begin{pmatrix}-\hat{x}\hat{y}\hat{z}(\psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z})\\ \hat{x}\hat{y}\hat{z}(\psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z})\\ 0\end{pmatrix} =$$

$$\begin{pmatrix}-\psi_y\hat{x}\hat{y}\hat{z} - \psi_{zx}\hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z}\\ \psi_x\hat{x}\hat{y}\hat{z} + \psi_{yz}\hat{x}\hat{y}\hat{z}\hat{x}\hat{y}\hat{z}\\ 0\end{pmatrix} = \begin{pmatrix}-\psi_y\hat{x}\hat{y}\hat{z} + \psi_{zx}\\ \psi_x\hat{x}\hat{y}\hat{z} - \psi_{yz}\\ 0\end{pmatrix} = \begin{pmatrix}\psi_{zx} - \psi_y\hat{x}\hat{y}\hat{z}\\ -\psi_{yz} + \psi_x\hat{x}\hat{y}\hat{z}\\ 0\end{pmatrix}$$

$\psi'_x = \psi_{zx}$
$\psi'_{yz} = -\psi_y$
$\psi'_y = -\psi_{yz}$
$\psi'_{zx} = \psi_x$
$\psi'_z = 0$
$\psi'_{xy} = 0$

And the ones that are not affected. For the following two, the same comments as for λ_1 apply:

$\psi'_{xyz} = \psi_{xyz}$
$\psi'_0 = \psi_0$

So, this means that the new ψ':

$$\psi' = \psi'_0 + \psi'_x\hat{x} + \psi'_y\hat{y} + \psi'_z\hat{z} + \psi'_{yz}\hat{y}\hat{z} + \psi'_{zx}\hat{z}\hat{x} + \psi'_{xy}\hat{x}\hat{y} + \psi'_{xyz}\hat{x}\hat{y}\hat{z}$$

Becomes:

$$\psi' = \psi_0 + \psi_{zx}\hat{x} - \psi_{yz}\hat{y} - \psi_y\hat{y}\hat{z} + \psi_x\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z} \quad (23)$$

In this case, we see that we have a change (complementary transformation/rotation?) where we have interchanged the axis $\hat{x}$ with the $\hat{z}\hat{x}$ plane and the axis $\hat{y}$ with the $\hat{y}\hat{z}$ plane.

5.3 Gell-Mann matrix λ_3

$$\psi' = \begin{pmatrix}\psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z}\\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z}\\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z}\end{pmatrix} = \lambda_3\psi = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z}\\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z}\\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z}\end{pmatrix} =$$

$$\begin{pmatrix}\psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z}\\ -\psi_y - \psi_{zx}\hat{x}\hat{y}\hat{z}\\ 0\end{pmatrix}$$

$\psi'_x = \psi_x$
$\psi'_{yz} = \psi_{yz}$
$\psi'_y = -\psi_y$
$\psi'_{zx} = -\psi_{zx}$
$\psi'_z = 0$
$\psi'_{xy} = 0$

And the ones that are not affected. For the following two, the same comments as for λ_1 apply:

$\psi'_{xyz} = \psi_{xyz}$
$\psi'_0 = \psi_0$

So, this means that the new ψ':

$$\psi' = \psi'_0 + \psi'_x\hat{x} + \psi'_y\hat{y} + \psi'_z\hat{z} + \psi'_{yz}\hat{y}\hat{z} + \psi'_{zx}\hat{z}\hat{x} + \psi'_{xy}\hat{x}\hat{y} + \psi'_{xyz}\hat{x}\hat{y}\hat{z}$$

Becomes:

$$\psi' = \psi_0 + \psi_x\hat{x} - \psi_y\hat{y} + \psi_{yz}\hat{y}\hat{z} - \psi_{zx}\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z} \quad (24)$$

5.4 Gell-Mann matrix λ_4

$$\psi' = \begin{pmatrix}\psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z}\\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z}\\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z}\end{pmatrix} = \lambda_4\psi = \begin{pmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z}\\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z}\\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z}\end{pmatrix} =$$

$$\begin{pmatrix}\psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z}\\ 0\\ \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z}\end{pmatrix}$$

$\psi'_x = \psi_z$
$\psi'_{yz} = \psi_{xy}$
$\psi'_y = 0$
$\psi'_{zx} = 0$
$\psi'_z = \psi_x$
$\psi'_{xy} = \psi_{yz}$

And the ones that are not affected. For the following two, the same comments as for λ_1 apply:

$\psi'_{xyz} = \psi_{xyz}$
$\psi'_0 = \psi_0$

So, this means that the new ψ':

$$\psi' = \psi'_0 + \psi'_x\hat{x} + \psi'_y\hat{y} + \psi'_z\hat{z} + \psi'_{yz}\hat{y}\hat{z} + \psi'_{zx}\hat{z}\hat{x} + \psi'_{xy}\hat{x}\hat{y} + \psi'_{xyz}\hat{x}\hat{y}\hat{z}$$

Becomes:

$$\psi' = \psi_0 + \psi_z\hat{x} + \psi_x\hat{z} + \psi_{xy}\hat{y}\hat{z} + \psi_{yz}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z} \quad (25)$$

We see an interchange between axes x and z and the planes yz and xy.

5.5 Gell-Mann matrix λ_5

$$\psi' = \begin{pmatrix}\psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z}\\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z}\\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z}\end{pmatrix} = \lambda_5\psi = \begin{pmatrix}0 & 0 & -i\\ 0 & 0 & 0\\ i & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z}\\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z}\\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z}\end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & -\hat{x}\hat{y}\hat{z} \\ 0 & 0 & 0 \\ \hat{x}\hat{y}\hat{z} & 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix} = \begin{pmatrix} -\hat{x}\hat{y}\hat{z}(\psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z}) \\ 0 \\ \hat{x}\hat{y}\hat{z}(\psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z}) \end{pmatrix} =$$

$$\begin{pmatrix} -\hat{x}\hat{y}\hat{z}\psi_z + \psi_{xy} \\ 0 \\ \hat{x}\hat{y}\hat{z}\psi_x - \psi_{yz} \end{pmatrix} = \begin{pmatrix} \psi_{xy} - \psi_z\hat{x}\hat{y}\hat{z} \\ 0 \\ -\psi_{yz} + \psi_x\hat{x}\hat{y}\hat{z} \end{pmatrix}$$

$\psi'_x = \psi_{xy}$
$\psi'_{yz} = -\psi_z$
$\psi'_y = 0$
$\psi'_{zx} = 0$
$\psi'_z = -\psi_{yz}$
$\psi'_{xy} = \psi_x$

And the ones that are not affected. For the following two, the same comments as for λ_1 apply:

$\psi'_{xyz} = \psi_{xyz}$
$\psi'_0 = \psi_0$

So, this means that the new ψ':

$$\psi' = \psi'_0 + \psi'_x\hat{x} + \psi'_y\hat{y} + \psi'_z\hat{z} + \psi'_{yz}\hat{y}\hat{z} + \psi'_{zx}\hat{z}\hat{x} + \psi'_{xy}\hat{x}\hat{y} + \psi'_{xyz}\hat{x}\hat{y}\hat{z}$$

Becomes:

$$\psi' = \psi_0 + \psi_{xy}\hat{x} - \psi_{yz}\hat{z} - \psi_z\hat{y}\hat{z} + \psi_x\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z} \quad (26)$$

5.6 Gell-Mann matrix λ_6

$$\psi' = \begin{pmatrix} \psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z} \\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z} \\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix} = \lambda_6\psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix} =$$

$$\begin{pmatrix} 0 \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \end{pmatrix}$$

$\psi'_x = 0$
$\psi'_{yz} = 0$
$\psi'_y = \psi_z$
$\psi'_{zx} = \psi_{xy}$
$\psi'_z = \psi_y$
$\psi'_{xy} = \psi_{zx}$

And the ones that are not affected. For the following two, the same comments as for λ_1 apply:

$\psi'_{xyz} = \psi_{xyz}$
$\psi'_0 = \psi_0$

So, this means that the new ψ':

$$\psi' = \psi'_0 + \psi'_x\hat{x} + \psi'_y\hat{y} + \psi'_z\hat{z} + \psi'_{yz}\hat{y}\hat{z} + \psi'_{zx}\hat{z}\hat{x} + \psi'_{xy}\hat{x}\hat{y} + \psi'_{xyz}\hat{x}\hat{y}\hat{z}$$

Becomes:

$$\psi' = \psi_0 + \psi_z\hat{y} + \psi_y\hat{z} + \psi_{xy}\hat{z}\hat{x} + \psi_{zx}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z} \quad (27)$$

5.7 Gell-Mann matrix λ_7

$$\psi' = \begin{pmatrix} \psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z} \\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z} \\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix} = \lambda_7\psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}\begin{pmatrix} \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\hat{x}\hat{y}\hat{z} \\ 0 & \hat{x}\hat{y}\hat{z} & 0 \end{pmatrix}\begin{pmatrix} \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix} = \begin{pmatrix} 0 \\ -\hat{x}\hat{y}\hat{z}(\psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z}) \\ \hat{x}\hat{y}\hat{z}(\psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z}) \end{pmatrix} =$$

$$\begin{pmatrix} 0 \\ -\hat{x}\hat{y}\hat{z}\psi_z + \psi_{xy} \\ \hat{x}\hat{y}\hat{z}\psi_y - \psi_{zx} \end{pmatrix} = \begin{pmatrix} 0 \\ \psi_{xy} - \psi_z\hat{x}\hat{y}\hat{z} \\ -\psi_{zx} + \psi_y\hat{x}\hat{y}\hat{z} \end{pmatrix}$$

$\psi'_x = 0$
$\psi'_{yz} = 0$
$\psi'_y = \psi_{xy}$
$\psi'_{zx} = -\psi_z$
$\psi'_z = -\psi_{zx}$
$\psi'_{xy} = \psi_y$

And the ones that are not affected. For the following two, the same comments as for λ_1 apply:

$\psi'_{xyz} = \psi_{xyz}$
$\psi'_0 = \psi_0$

So, this means that the new ψ':

$$\psi' = \psi'_0 + \psi'_x\hat{x} + \psi'_y\hat{y} + \psi'_z\hat{z} + \psi'_{yz}\hat{y}\hat{z} + \psi'_{zx}\hat{z}\hat{x} + \psi'_{xy}\hat{x}\hat{y} + \psi'_{xyz}\hat{x}\hat{y}\hat{z}$$

Becomes:

$$\psi' = \psi_0 + \psi_{xy}\hat{y} - \psi_{zx}\hat{z} - \psi_z\hat{z}\hat{x} + \psi_y\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z} \quad (28)$$

5.8 Gell-Mann matrix λ_8

$$\psi' = \begin{pmatrix} \psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z} \\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z} \\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix} = \lambda_4\psi = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\begin{pmatrix} \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{1}{\sqrt{3}}\psi_x + \frac{1}{\sqrt{3}}\psi_{yz}\hat{x}\hat{y}\hat{z} \\ \frac{1}{\sqrt{3}}\psi_y + \frac{1}{\sqrt{3}}\psi_{zx}\hat{x}\hat{y}\hat{z} \\ -\frac{2}{\sqrt{3}}\psi_z - \frac{2}{\sqrt{3}}\psi_{xy}\hat{x}\hat{y}\hat{z} \end{pmatrix}$$

$\psi'_x = \frac{1}{\sqrt{3}}\psi_x$
$\psi'_{yz} = \frac{1}{\sqrt{3}}\psi_{yz}$
$\psi'_{yz} = \frac{1}{\sqrt{3}}\psi_{yz}$
$\psi'_{zx} = \frac{1}{\sqrt{3}}\psi_{zx}$
$\psi'_z = -\frac{2}{\sqrt{3}}\psi_z$
$\psi'_{xy} = -\frac{2}{\sqrt{3}}\psi_{xy}$

And the ones that are not affected. For the following two, the same comments as for λ_1 apply:

$\psi'_{xyz} = \psi_{xyz}$
$\psi'_0 = \psi_0$

So, this means that the new ψ':

$$\psi' = \psi'_0 + \psi'_x\hat{x} + \psi'_y\hat{y} + \psi'_z\hat{z} + \psi'_{yz}\hat{y}\hat{z} + \psi'_{zx}\hat{z}\hat{x} + \psi'_{xy}\hat{x}\hat{y} + \psi'_{xyz}\hat{x}\hat{y}\hat{z}$$

Becomes:

$$\psi' = \psi_0 + \frac{1}{\sqrt{3}}\psi_x\hat{x} + \frac{1}{\sqrt{3}}\psi_y\hat{y} - \frac{2}{\sqrt{3}}\psi_z\hat{z} + \frac{1}{\sqrt{3}}\psi_{yz}\hat{y}\hat{z} + \frac{1}{\sqrt{3}}\psi_{zx}\hat{z}\hat{x} - \frac{2}{\sqrt{3}}\psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z} \quad (29)$$

Here the square roots of three are used by Gell-Mann to keep the same norm to the final spinor ψ' independently of the transformation, as the rest of transformations (λ_1- λ_7) have less elements in the result.

This is another hint that clearly, he did not consider ψ_0 and ψ_{xyz} in the transformations and probably the outcome value of them should be zero. Another hint is that in most of the transformations, some other elements become zero. If we consider the transformations as rotations, these zeros come probably form the "hidden ψ_0 and ψ_{xyz} that are already zero" in the original ψ.

But this is too detailed. Let's go to the summary.

6. Summary of application of Gell-Mann matrices

In the previous chapter we have obtained the results of applying the Gell-Mann matrices to ψ (22)-(29) in geometric algebra notation.

Considering this is the original ψ:

$$\psi = \psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$

The new ψ' obtained when applying each of the Gell-Mann matrices λ_i is:

$\psi' = (\lambda_1 \to \psi) = \psi_0 + \psi_y\hat{x} + \psi_x\hat{y} + \psi_{zx}\hat{y}\hat{z} + \psi_{yz}\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$
$\psi' = (\lambda_2 \to \psi) = \psi_0 + \psi_{zx}\hat{x} - \psi_{yz}\hat{y} - \psi_y\hat{y}\hat{z} + \psi_x\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$
$\psi' = (\lambda_3 \to \psi) = \psi_0 + \psi_x\hat{x} - \psi_y\hat{y} + \psi_{yz}\hat{y}\hat{z} - \psi_{zx}\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$
$\psi' = (\lambda_4 \to \psi) = \psi_0 + \psi_z\hat{x} + \psi_x\hat{z} + \psi_{xy}\hat{y}\hat{z} + \psi_{yz}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$
$\psi' = (\lambda_5 \to \psi) = \psi_0 + \psi_{xy}\hat{x} - \psi_{yz}\hat{z} - \psi_z\hat{y}\hat{z} + \psi_x\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$
$\psi' = (\lambda_6 \to \psi) = \psi_0 + \psi_z\hat{y} + \psi_y\hat{z} + \psi_{xy}\hat{z}\hat{x} + \psi_{zx}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$
$\psi' = (\lambda_7 \to \psi) = \psi_0 + \psi_{xy}\hat{y} - \psi_{zx}\hat{z} - \psi_z\hat{z}\hat{x} + \psi_y\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$
$\psi' = (\lambda_8 \to \psi) = \psi_0 + \frac{1}{\sqrt{3}}\psi_x\hat{x} + \frac{1}{\sqrt{3}}\psi_y\hat{y} - \frac{2}{\sqrt{3}}\psi_z\hat{z} + \frac{1}{\sqrt{3}}\psi_{yz}\hat{y}\hat{z} + \frac{1}{\sqrt{3}}\psi_{zx}\hat{z}\hat{x} - \frac{2}{\sqrt{3}}\psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$

7. Using Gell-Mann matrices of 4x4

As commented in chapter 4, there is another way of solving the issue that the Gell-Mann matrices are 3x3 and the spinor matrix vector has four rows. Before, we have solved it eliminating the last row (with an explanation of why this is possible). In this chapter, we will show how to solve it the opposite way. This is, using Gell-Mann matrices of 4x4. If we take the example of λ_1.

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We can add a new row and column with all zeros except a 1 in the diagonal

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We can see that we will obtain the same result as in chapter in 5.1. If we consider ψ and ψ' as:

$$\psi = \begin{pmatrix} \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \\ \psi_{xyz} - \psi_0\hat{x}\hat{y}\hat{z} \end{pmatrix}$$

$$\psi' = \begin{pmatrix} \psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z} \\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z} \\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z} \\ \psi'_{xyz} - \psi'_0\hat{x}\hat{y}\hat{z} \end{pmatrix}$$

We multiply:

$$\psi' = \begin{pmatrix} \psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z} \\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z} \\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z} \\ \psi'_{xyz} - \psi'_0\hat{x}\hat{y}\hat{z} \end{pmatrix} = \lambda_1\psi =$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \\ \psi_{xyz} - \psi_0\hat{x}\hat{y}\hat{z} \end{pmatrix} = \begin{pmatrix} \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ 0 \\ \psi_{xyz} - \psi_0\hat{x}\hat{y}\hat{z} \end{pmatrix}$$

And we get the same result as in 5.1.

$\psi'_x = \psi_y$
$\psi'_{yz} = \psi_{zx}$
$\psi'_y = \psi_x$
$\psi'_{zx} = \psi_{yz}$
$\psi'_z = 0$
$\psi'_{xy} = 0$
$\psi'_{xyz} = \psi_{xyz}$
$\psi'_0 = \psi_0$

Leading to:
As in chapter 5.1

$\psi' = \psi_0 + \psi_y\hat{x} + \psi_x\hat{y} + \psi_{zx}\hat{y}\hat{z} + \psi_{yz}\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$

There is another option, as already commented. That the Gell-Mann matrix λ_1 it affects also ψ_0 and ψ_{xyz} converting them to zero. That would correspond to:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Multiplying:

$$\psi' = \begin{pmatrix} \psi'_x + \psi'_{yz}\hat{x}\hat{y}\hat{z} \\ \psi'_y + \psi'_{zx}\hat{x}\hat{y}\hat{z} \\ \psi'_z + \psi'_{xy}\hat{x}\hat{y}\hat{z} \\ \psi'_{xyz} - \psi'_0\hat{x}\hat{y}\hat{z} \end{pmatrix} = \lambda_1\psi =$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xy}\hat{x}\hat{y}\hat{z} \\ \psi_{xyz} - \psi_0\hat{x}\hat{y}\hat{z} \end{pmatrix} = \begin{pmatrix} \psi_y + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_x + \psi_{yz}\hat{x}\hat{y}\hat{z} \\ 0 \\ 0 \end{pmatrix}$$

And we will get the same result as in chapter 5.1:

$\psi'_x = \psi_y$
$\psi'_{yz} = \psi_{zx}$
$\psi'_y = \psi_x$
$\psi'_{zx} = \psi_{yz}$
$\psi'_z = 0$
$\psi'_{xy} = 0$
$\psi'_{xyz} = 0$
$\psi'_0 = 0$

Leading to:

$\psi' = \psi_y\hat{x} + \psi_x\hat{y} + \psi_{zx}\hat{y}\hat{z} + \psi_{yz}\hat{z}\hat{x}$

As commented, the Gell-Mann matrices do not even consider ψ_0 and ψ_{xyz} so it is not possible to know, which one is correct. Although to keep the rotation symmetry seems more plausible that they really become zero.

There are also other possibilities, in general,

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 & a_{14} \\ 1 & 0 & 0 & a_{24} \\ 0 & 0 & 0 & a_{34} \\ a_{14}^* & a_{24}^* & a_{34}^* & a_{44} \end{pmatrix}$$

Where the asterisk means conjugate (whether it is in *i* or in xyz notation). And a_{44} is real.

That would mean that another transform/rotation is applied to ψ_0 and ψ_{xyz} not consider in the original Gell-Mann transformations (could they be related to other forces?).

8. Possibilities to apply the Gell-Mann transformations using Geometric algebra operations (not matrices)

In chapter 6 you have a summary of the transformations obtained with Gell-Mann matrices, applied to a spinor in Geometric Algebra notation ψ to be converted to another one ψ'.

In Geometric Algebra the transformations are not normally performed that way. Instead, we use rotations or boosts to convert one multivector (in this case, representing a spinor) into another one.

In general, the transformation would something like this:

$$\psi' = \alpha_1 e^{-\frac{1}{2}\alpha_2\hat{x}} e^{-\frac{1}{2}\alpha_3\hat{y}} e^{-\frac{1}{2}\alpha_4\hat{z}} e^{-\frac{1}{2}\alpha_5\hat{y}\hat{z}} e^{-\frac{1}{2}\alpha_6\hat{z}\hat{x}} e^{-\frac{1}{2}\alpha_7\hat{x}\hat{y}} e^{-\frac{1}{2}\alpha_8\hat{x}\hat{y}\hat{z}} \psi e^{\frac{1}{2}\alpha_8\hat{x}\hat{y}\hat{z}} e^{\frac{1}{2}\alpha_7\hat{x}\hat{y}} e^{\frac{1}{2}\alpha_6\hat{z}\hat{x}} e^{\frac{1}{2}\alpha_5\hat{y}\hat{z}} e^{\frac{1}{2}\alpha_4\hat{z}} e^{\frac{1}{2}\alpha_3\hat{y}} e^{\frac{1}{2}\alpha_2\hat{x}}$$

Where ψ is the spinor multivector to which we want to make the transformation and ψ' the multivector obtained.

$\psi = \psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$

The exponentials to vectors (which square is +1) are boosts and can be also written as:

$e^{-\frac{1}{2}\alpha_2\hat{x}} = cosh\left(-\frac{1}{2}\alpha_2\right) + sinh\left(-\frac{1}{2}\alpha_2\right)\hat{x} = cosh\left(\frac{1}{2}\alpha_2\right) -$

$sinh\left(\frac{1}{2}\alpha_2\right)\hat{x}$

The exponentials to bivectors or to the trivector (which square is -1) are rotations of an angle α_i in the planes indicated by the bivector (or in the volume/time in the case of the trivector) and can be written also as [1,3]:

$$e^{-\frac{1}{2}\alpha_5\hat{y}\hat{z}} = cos\left(-\frac{1}{2}\alpha_5\right) + sin\left(-\frac{1}{2}\alpha_5\right)\hat{y}\hat{z} = cos\left(\frac{1}{2}\alpha_5\right) - sin\left(\frac{1}{2}\alpha_2\right)\hat{y}\hat{z}$$

If you do not understand what a rotation in the volume means, welcome to the club. But you can see an example of it in annex A.1 of [2] when I considered that the trivector (I still do) was somehow related to spin.

There is also a parameter α_1 that it is only a scalation. All α_i are real.

The important point is that we have also 8 α_i, so we have 8 degrees of freedom of transformations. This is the same as the 8 Gell-Mann matrices. So, a correspondence between them would be possible. Another thing is if it is necessary to find them, now that we have the table in chapter 6.

The same as commented above with rotations and boosts can be applied easier using a transformation as:

$$\psi' = (\beta_1 + \beta_2\hat{x} + \beta_3\hat{y} + \beta_4\hat{z} + \beta_5\hat{y}\hat{z} + \beta_6\hat{z}\hat{x} + \beta_7\hat{x}\hat{y} + \beta_8\hat{x}\hat{y}\hat{z})\psi$$

This is, to perform this pre-multiplication to ψ:

$$\psi' = (\beta_1 + \beta_2\hat{x} + \beta_3\hat{y} + \beta_4\hat{z} + \beta_5\hat{y}\hat{z} + \beta_6\hat{z}\hat{x} + \beta_7\hat{x}\hat{y} + \beta_8\hat{x}\hat{y}\hat{z})(\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z})$$

And even we could have the double-sided version as:

$$\psi' = (\delta_1 + \delta_2\hat{x} + \delta_3\hat{y} + \delta_4\hat{z} - \delta_5\hat{y}\hat{z} - \delta_6\hat{z}\hat{x} - \delta_7\hat{x}\hat{y} - \delta_8\hat{x}\hat{y}\hat{z})\psi(\delta_1 + \delta_2\hat{x} + \delta_3\hat{y} + \delta_4\hat{z} + \delta_5\hat{y}\hat{z} + \delta_6\hat{z}\hat{x} + \delta_7\hat{x}\hat{y} + \delta_8\hat{x}\hat{y}\hat{z})$$

In both cases, we have 8 parameters that modify the ψ function. So, again there could be found the β_i or δ_i that creates the same transformations as indicated in chapter 6.

9. Bra-ket products using spinor multivector

Before showing how the bra-ket multiplication is, I have two show the operation called reversion of a multivector. This operation reverses the order of the bivectors and trivector.

It is the equivalent of the conjugate of a complex number applied to a multivector.

In fact, in the case of Geometric Algebra $Cl_{3,0}$ (the one we have used along the paper), what it does is to change the sign of the coefficients multiplying the bivectors and the trivector (the ones that have square -1). And it keeps the same the coefficients of the scalar and the vectors.

This is, if we have a multivector ψ:

$$\psi = \psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$$

Its reverse is:

$$\psi^\dagger = \psi^* = \psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{z}\hat{y} + \psi_{zx}\hat{x}\hat{z} + \psi_{xy}\hat{y}\hat{x} + \psi_{xyz}\hat{z}\hat{y}\hat{x} = \psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} - \psi_{yz}\hat{y}\hat{z} - \psi_{zx}\hat{z}\hat{x} - \psi_{xy}\hat{x}\hat{y} - \psi_{xyz}\hat{x}\hat{y}\hat{z}$$

In Euclidean geometry $Cl_{3,0}$, the reverse $\psi^\dagger$ and the conjugate ψ^* are the same thing. In non-Euclidean metric (more specifically in non-orthogonal metric), we will see that this not hold.

The bra-ket product is defined as:

$$\psi^\dagger\psi = \psi^*\psi =$$
$$= (\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{z}\hat{y} + \psi_{zx}\hat{x}\hat{z} + \psi_{xy}\hat{y}\hat{x} + \psi_{xyz}\hat{z}\hat{y}\hat{x})(\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}) =$$
$$= (\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} - \psi_{yz}\hat{y}\hat{z} - \psi_{zx}\hat{z}\hat{x} - \psi_{xy}\hat{x}\hat{y} - \psi_{xyz}\hat{x}\hat{y}\hat{z})(\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z})$$

And the important thing is that if we perform the complete operation we get (see Annex A1):

$$\psi^\dagger\psi = \psi^*\psi = \rho + \vec{j} \quad (29.1)$$

Being ρ the probability density (positive defined):

$$\rho = {\psi_0}^2 + {\psi_x}^2 + {\psi_y}^2 + {\psi_z}^2 + {\psi_{yz}}^2 + {\psi_{zx}}^2 + {\psi_{xy}}^2 + {\psi_{xyz}}^2 \quad (29.2)$$

And $\vec{j}$ is the fermionic current as defined in [48]:

$$\vec{j} = 2(\psi_0\psi_y + \psi_x\psi_{xy} - \psi_z\psi_{yz} + \psi_{zx}\psi_{xyz})\hat{x} + 2(\psi_0\psi_y + \psi_x\psi_{xy} - \psi_z\psi_{yz} + \psi_{zx}\psi_{xyz})\hat{y} + 2(\psi_0\psi_z - \psi_x\psi_{zx} + \psi_y\psi_{yz} + \psi_{xy}\psi_{xyz})\hat{z} \quad (29.3)$$

All the coefficients in bivectors and the trivector cancel as you can see in Annexes A1 and A2.

So, this means, that the spinor multivector is not only a "cool mathematical artifact". You can obtain the same information as with the typical spinor matrix vectors and in a much simpler way. See Annex A1 to see what I am talking about. We just have to perform a multiplication. And in Annex A2, you can check that the result using Geometric Algenra is exactly the same as using matrix algebra, so all the process is validated [54].

10. Bra-ket product in non-Euclidean metric (under gravitational effects)

In general, in non-Euclidean metric (non-orthonormal and non-orthogonal) the following equations apply for the basis vectors (instead of the equation 8.1 in chapter 2) [2,47]:

$$\begin{aligned} \hat{x}^2 &= \|\hat{x}\|^2 = g_{xx} \\ \hat{y}^2 &= \|\hat{y}\|^2 = g_{yy} \\ \hat{z}^2 &= \|\hat{z}\|^2 = g_{zz} \\ \hat{x}\hat{y} &= 2g_{xy} - \hat{y}\hat{x} \\ \hat{y}\hat{z} &= 2g_{yz} - \hat{z}\hat{y} \\ \hat{z}\hat{x} &= 2g_{zx} - \hat{x}\hat{z} \end{aligned} \quad (30)$$

In the case of non-orthonormal but yes orthogonal (diagonal) metric, these equations are simplified as:

$$\begin{aligned} \hat{x}^2 &= \|\hat{x}\|^2 = g_{xx} \\ \hat{y}^2 &= \|\hat{y}\|^2 = g_{yy} \\ \hat{z}^2 &= \|\hat{z}\|^2 = g_{zz} \\ \hat{x}\hat{y} &= -\hat{y}\hat{x} \\ \hat{y}\hat{z} &= -\hat{z}\hat{y} \\ \hat{z}\hat{x} &= -\hat{x}\hat{z} \end{aligned} \quad (31)$$

The bra-ket product in non-orthonormal but yes orthogonal metric leads again to (see Annex A3):

$$\psi^\dagger\psi = \psi^*\psi = \rho + \vec{j} \quad (32)$$

But in this case:

$$\rho = {\psi_0}^2 + {\psi_x}^2 g_{xx} + {\psi_y}^2 g_{yy} + {\psi_z}^2 g_{zz} + {\psi_{yz}}^2 g_{yy}g_{zz} + {\psi_{zx}}^2 g_{zz}g_{xx} + {\psi_{xy}}^2 g_{xx}g_{yy} + {\psi_{xyz}}^2 g_{xx}g_{yy}g_{zz} \quad (33)$$

And:

$$\vec{j} = 2(\psi_0\psi_x - \psi_y\psi_{xy}g_{yy} + \psi_z\psi_{zx}g_{zz} + \psi_{yz}\psi_{xyz}g_{yy}g_{zz})\hat{x} + 2(+\psi_0\psi_y + \psi_x\psi_{xy}g_{xx} - \psi_{yz}\psi_z g_{zz} + \psi_{zx}\psi_{xyz}g_{zz}g_{xx})9\hat{y} + 2(+\psi_0\psi_z - \psi_x\psi_{zx}g_{xx} + \psi_y\psi_{yz}g_{yy} + \psi_{xy}\psi_{xyz}g_{xx}g_{yy})\hat{z} \quad (34)$$

For the case of non-orthonormal and non-orthogonal case (equations 30), you can check that nightmare in Annex A4 and never come back.

11. Effects of gravitation in muon g-2 issue

Normally, it is said that the gravitational effects are too small to affect the g-2 issue. I will show that this could not be correct.

If you check equation:

$$\rho = {\psi_0}^2 + {\psi_x}^2 g_{xx} + {\psi_y}^2 g_{yy} + {\psi_z}^2 g_{zz} + {\psi_{yz}}^2 g_{yy}g_{zz} + {\psi_{zx}}^2 g_{zz}g_{xx} +$$

$\psi_{xy}{}^2 g_{xx} g_{yy} + \psi_{xyz}{}^2 g_{xx} g_{yy} g_{zz}$

And we consider that we are in the surface of earth. We put the x axis in the radial direction, and y and z perpendicular to it. According Schwarzschild metric, the effect in axis x (radial) would be:

$g_{xx} = g_{rr} = \left(1 - \frac{GM}{c^2 r}\right)^{-1}$

Where:

$G = 6{,}6743E - \frac{11Nm^2}{kg}$

$M_{earth} = 5{,}9722E24\ kg$

$r = r_{earth} = 6{,}371E6m$

This gives:

$g_{xx} = 1 + 1{,}392262E - 09$

And we can consider:

$g_{yy} = g_{zz} = 1$

We see in equation (33) that g_{xx} is included and is affecting 4 of the 8 elements of the equation. So, the probability (and therefore the probability of the possible states that the particle can take) can be impacted by it (it is not something neglectable that only affects one of the 8 elements)

But can this small value in equation (34) affect something?

If we go to the g-2 experiment, we have [55]:

Theoretical value:

$a_t = \frac{g-2}{2} = 1{,}1659181E - 03$

Traditionally measured value:

$a_m = \frac{g-2}{2} = 1{,}16592089E - 03$

If we get the difference between the values we have:

$a_m - a_t = 2{,}79E - 09$

If we divide this value with the "added element" in (34) we have:

$\frac{2{,}79E-09}{1{,}392262E-09} \cong 2$

This means, they are in the same order. Perfectly the g_{xx} value could be affecting the measurement. As commented g_{xx} appears in 4 of the 8 elements of the multivector (33). It is not only affecting in one direction.

Even taking the latest value measured in Fermilab, we are in the same order [56]:

$a_{fermilab} = \frac{g-2}{2} = 1{,}16592055E - 03$

$a_{fermilab} - a_t = 2{,}45E - 09$

$\frac{2{,}45E-09}{1{,}392262E-09} \cong 1{,}76$

Again, in the same order of magnitude. So, we can check, that yes, the gravitational effects creating a non-Euclidean metric could have their effects in muon g-2 effect.

CONCLUSION

We have found a way of applying the Gell-Mann transformations made by the λ_i matrices using Geometric Algebra $Cl_{3,0}$. And without the need of adding the time as an ad-hoc dimension, but just considering that:

$\hat{\boldsymbol{t}} = \hat{\boldsymbol{x}}\hat{\boldsymbol{y}}\hat{\boldsymbol{z}}$

The transformations are as follows. Considering the original ψ:

$\psi = \psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$

The new ψ' obtained when applying each of the Gell-Mann matrices λ_i is:

$\psi' = (\lambda_1 \to \psi) = \psi_0 + \psi_y\hat{x} + \psi_x\hat{y} + \psi_{zx}\hat{y}\hat{z} + \psi_{yz}\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$

$\psi' = (\lambda_2 \to \psi) = \psi_0 + \psi_{zx}\hat{x} - \psi_{yz}\hat{y} - \psi_y\hat{y}\hat{z} + \psi_x\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$

$\psi' = (\lambda_3 \to \psi) = \psi_0 + \psi_x\hat{x} - \psi_y\hat{y} + \psi_{yz}\hat{y}\hat{z} - \psi_{zx}\hat{z}\hat{x} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$

$\psi' = (\lambda_4 \to \psi) = \psi_0 + \psi_z\hat{x} + \psi_x\hat{z} + \psi_{xy}\hat{y}\hat{z} + \psi_{yz}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$

$\psi' = (\lambda_5 \to \psi) = \psi_0 + \psi_{xy}\hat{x} - \psi_{yz}\hat{z} - \psi_z\hat{y}\hat{z} + \psi_x\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$

$\psi' = (\lambda_6 \to \psi) = \psi_0 + \psi_z\hat{y} + \psi_y\hat{z} + \psi_{xy}\hat{z}\hat{x} + \psi_{zx}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$

$\psi' = (\lambda_7 \to \psi) = \psi_0 + \psi_{xy}\hat{y} - \psi_{zx}\hat{z} - \psi_z\hat{z}\hat{x} + \psi_y\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$

$\psi' = (\lambda_8 \to \psi) = \psi_0 + \frac{1}{\sqrt{3}}\psi_x\hat{x} + \frac{1}{\sqrt{3}}\psi_y\hat{y} - \frac{2}{\sqrt{3}}\psi_z\hat{z} + \frac{1}{\sqrt{3}}\psi_{yz}\hat{y}\hat{z} + \frac{1}{\sqrt{3}}\psi_{zx}\hat{z}\hat{x} - \frac{2}{\sqrt{3}}\psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}$

Taking into account that Gell-Mann matrices do not consider at all the existence of ψ_0 and ψ_{xyz} , it is possible that we should consider these two elements zero from the beginning. Anyhow, above relations would correspond to the most general case.

We have also worked in the bra-ket product using geometric algebra. For the Euclidean case we have the equation (where the cross sign means reverse and the asterisk means conjugate, both mean the same in $Cl_{3,0}$):

$\psi^\dagger\psi = \psi^*\psi =$

$= (\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{z}\hat{y} + \psi_{zx}\hat{x}\hat{z} + \psi_{xy}\hat{y}\hat{x} + \psi_{xyz}\hat{z}\hat{y}\hat{x})(\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}) =$

$= (\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} - \psi_{yz}\hat{y}\hat{z} - \psi_{zx}\hat{z}\hat{x} - \psi_{xy}\hat{x}\hat{y} - \psi_{xyz}\hat{x}\hat{y}\hat{z})(\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}) = \rho + \vec{j}$

Being ρ the probability density:

$\rho = \psi_0{}^2 + \psi_x{}^2 + \psi_y{}^2 + \psi_z{}^2 + \psi_{yz}{}^2 + \psi_{zx}{}^2 + \psi_{xy}{}^2 + \psi_{xyz}{}^2$

And $\vec{j}$ the fermionic current:

$\vec{j} = 2(\psi_0\psi_y + \psi_x\psi_{xy} - \psi_z\psi_{yz} + \psi_{zx}\psi_{xyz})\hat{x} + 2(\psi_0\psi_y + \psi_x\psi_{xy} - \psi_z\psi_{yz} + \psi_{zx}\psi_{xyz})\hat{y} + 2(\psi_0\psi_z - \psi_x\psi_{zx} + \psi_y\psi_{yz} + \psi_{xy}\psi_{xyz})\hat{z}$

We have made the same in the case of orthogonal but not orthonormal metric, leading to:

$\psi^\dagger\psi = \psi^*\psi = \rho + \vec{j}$

But in this case:

$\rho = \psi_0{}^2 + \psi_x{}^2 g_{xx} + \psi_y{}^2 g_{yy} + \psi_z{}^2 g_{zz} + \psi_{yz}{}^2 g_{yy} g_{zz} + \psi_{zx}{}^2 g_{zz} g_{xx} + \psi_{xy}{}^2 g_{xx} g_{yy} + \psi_{xyz}{}^2 g_{xx} g_{yy} g_{zz}$

And:

$\vec{j} = 2(\psi_0\psi_x - \psi_y\psi_{xy} g_{yy} + \psi_z\psi_{zx} g_{zz} + \psi_{yz}\psi_{xyz} g_{yy} g_{zz})\hat{x} + 2(+\psi_0\psi_y + \psi_x\psi_{xy} g_{xx} - \psi_{yz}\psi_z g_{zz} + \psi_{zx}\psi_{xyz} g_{zz} g_{xx})\hat{y} + 2(+\psi_0\psi_z - \psi_x\psi_{zx} g_{xx} + \psi_y\psi_{yz} g_{yy} + \psi_{xy}\psi_{xyz} g_{xx} g_{yy})\hat{z}$

We have shown also that the g-2 issue of the muon could be perfectly related to gravitational issues (to non-Euclidean space). The difference of the values of g-2 of the muon are:

$a_m - a_t = 2{,}79E - 09$

And the effect of the non-Euclidean metric on the surface of Earth is:

$g_{xx} - 1 = 1{,}392262E - 09$

As we can check, they are in the same order, being one approximately 2 times the other. So, gravitational effects could indeed affect the g-2 value of the muon on the surface or Earth as commented.

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ANNEX A1

Annex A1. Bra-Ket product in Euclidean metric

If we operate:

$\psi^*\psi = (\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} - \psi_{yz}\hat{y}\hat{z} - \psi_{zx}\hat{z}\hat{x} - \psi_{xy}\hat{x}\hat{y} - \psi_{xyz}\hat{x}\hat{y}\hat{z})(\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}) =$

${\psi_0}^2 + \psi_0\psi_x\hat{x} + \psi_0\psi_y\hat{y} + \psi_0\psi_z\hat{z} + \psi_0\psi_{yz}\hat{y}\hat{z} + \psi_0\psi_{zx}\hat{z}\hat{x} + \psi_0\psi_{xy}\hat{x}\hat{y} + \psi_0\psi_{xyz}\hat{x}\hat{y}\hat{z} +$

$\psi_x\psi_0\hat{x} + {\psi_x}^2 + \psi_x\psi_y\hat{x}\hat{y} - \psi_x\psi_z\hat{z}\hat{x} + \psi_x\psi_{yz}\hat{x}\hat{y}\hat{z} - \psi_x\psi_{zx}\hat{z} + \psi_x\psi_{xy}\hat{y} + \psi_x\psi_{xyz}\hat{y}\hat{z} +$

$\psi_y\psi_0\hat{y} - \psi_y\psi_x\hat{x}\hat{y} + {\psi_y}^2 + \psi_y\psi_z\hat{y}\hat{z} + \psi_y\psi_{yz}\hat{z} + \psi_y\psi_{zx}\hat{x}\hat{y}\hat{z} - \psi_y\psi_{xy}\hat{x} + \psi_y\psi_{xyz}z\hat{x} +$

$\psi_z\psi_0\hat{z} + \psi_z\psi_x\hat{z}\hat{x} - \psi_z\psi_y\hat{y}\hat{z} + {\psi_z}^2 - \psi_z\psi_{yz}\hat{y} + \psi_z\psi_{zx}\hat{x} + \psi_z\psi_{xy}\hat{x}\hat{y}\hat{z} + \psi_z\psi_{xyz}\hat{x}\hat{y}$

$-\psi_{yz}\psi_0\hat{y}\hat{z} - \psi_{yz}\psi_x\hat{x}\hat{y}\hat{z} + \psi_{yz}\psi_y\hat{z} - \psi_{yz}\psi_z\hat{y} + {\psi_{yz}}^2 + \psi_{yz}\psi_{zx}\hat{x}\hat{y} - \psi_{yz}\psi_{xy}\hat{z}\hat{x} + \psi_{yz}\psi_{xyz}\hat{x} -$

$-\psi_{zx}\psi_0\hat{z}\hat{x} - \psi_{zx}\psi_x\hat{z} - \psi_{zx}\psi_y\hat{x}\hat{y}\hat{z} + \psi_{zx}\psi_z\hat{x} - \psi_{zx}\psi_{yz}\hat{x}\hat{y} + {\psi_{zx}}^2 + \psi_{zx}\psi_{xy}\hat{y}\hat{z} + \psi_{zx}\psi_{xyz}\hat{y} -$

$-\psi_{xy}\psi_0\hat{x}\hat{y} + \psi_{xy}\psi_x\hat{y} - \psi_{xy}\psi_y\hat{x} - \psi_{xy}\psi_z\hat{x}\hat{y}\hat{z} + \psi_{xy}\psi_{yz}\hat{z}\hat{x} - \psi_{xy}\psi_{zx}\hat{y}\hat{z} + {\psi_{xy}}^2 + \psi_{xy}\psi_{xyz}\hat{z} -$

$-\psi_{xyz}\psi_0\hat{x}\hat{y}\hat{z} - \psi_{xyz}\psi_x\hat{y}\hat{z} - \psi_{xyz}\psi_y\hat{z}\hat{x} - \psi_{xyz}\psi_z\hat{x}\hat{y} + \psi_{xyz}\psi_{yz}\hat{x} + \psi_{xyz}\psi_{zx}\hat{y} + \psi_{xyz}\psi_{xy}\hat{z} + {\psi_{xyz}}^2$

And now if we take only the scalars we get:

${\psi_0}^2 + {\psi_x}^2 + {\psi_y}^2 + {\psi_z}^2 + {\psi_{yz}}^2 + {\psi_{zx}}^2 + {\psi_{xy}}^2 + {\psi_{xyz}}^2$

We will call this sum ρ:

$\rho = {\psi_0}^2 + {\psi_x}^2 + {\psi_y}^2 + {\psi_z}^2 + {\psi_{yz}}^2 + {\psi_{zx}}^2 + {\psi_{xy}}^2 + {\psi_{xyz}}^2$

If we separate the components that multiply by $\hat{x}$ we get:

$\psi_0\psi_x + \psi_x\psi_0 - \psi_y\psi_{xy} + \psi_z\psi_{zx} + \psi_{yz}\psi_{xyz} + \psi_{zx}\psi_z - \psi_{xy}\psi_y + \psi_{xyz}\psi_{yz} = 2(\psi_x\psi_0 - \psi_y\psi_{xy} + \psi_z\psi_{zx} + \psi_{yz}\psi_{xyz})$

In $\hat{y}$ we get:

$\psi_0\psi_y + \psi_x\psi_{xy} + \psi_y\psi_0 - \psi_z\psi_{yz} - \psi_{yz}\psi_z + \psi_{zx}\psi_{xyz} + \psi_{xy}\psi_x + \psi_{xyz}\psi_{zx} = 2(\psi_0\psi_y + \psi_x\psi_{xy} - \psi_z\psi_{yz} + \psi_{zx}\psi_{xyz})$

In $\hat{z}$ we get:

$\psi_0\psi_z - \psi_x\psi_{zx} + \psi_y\psi_{yz} + \psi_z\psi_0 + \psi_{yz}\psi_y - \psi_{zx}\psi_x + \psi_{xy}\psi_{xyz} + \psi_{xyz}\psi_{xy} = 2(\psi_0\psi_z - \psi_x\psi_{zx} + \psi_y\psi_{yz} + \psi_{xy}\psi_{xyz})$

In $\hat{y}\hat{z}$:

$\psi_0\psi_z - \psi_x\psi_{zx} + \psi_y\psi_{yz} + \psi_z\psi_0 + \psi_{yz}\psi_y - \psi_{zx}\psi_x + \psi_{xy}\psi_{xyz} + \psi_{xyz}\psi_{xy} = 2(\psi_0\psi_z - \psi_x\psi_{zx} + \psi_y\psi_{yz} + \psi_{xy}\psi_{xyz})$

In $\hat{y}\hat{z}$:

$\psi_0\psi_{yz} + \psi_x\psi_{xyz} + \psi_y\psi_z - \psi_z\psi_y - \psi_{yz}\psi_0 + \psi_{zx}\psi_{xy} - \psi_{xy}\psi_{zx} - \psi_{xyz}\psi_x = 0$

In $\hat{z}\hat{x}$:

$\psi_0\psi_{zx} - \psi_x\psi_z + \psi_y\psi_{xyz} + \psi_z\psi_x - \psi_{yz}\psi_{xy} - \psi_{zx}\psi_0 + \psi_{xy}\psi_{yz} - \psi_{xyz}\psi_y = 0$

In $\hat{x}\hat{y}$:

$\psi_0\psi_{xy} + \psi_x\psi_y - \psi_y\psi_x + \psi_z\psi_{xyz} + \psi_{yz}\psi_{zx} - \psi_{zx}\psi_{yz} - \psi_{xy}\psi_0 - \psi_{xyz}\psi_z = 0$

In $\hat{x}\hat{y}\hat{z}$:

$\psi_0\psi_{xyz} + \psi_x\psi_{yz} + \psi_y\psi_{zx} + \psi_z\psi_{xy} - \psi_{yz}\psi_x - \psi_{zx}\psi_y - \psi_{xy}\psi_z - \psi_{xyz}\psi_0 = 0$

If we call vector $\vec{j}$ (fermionic current) the sum in $\hat{x}$, $\hat{y}$ and $\hat{z}$, we get:

$\vec{j} = 2(\psi_0\psi_y + \psi_x\psi_{xy} - \psi_z\psi_{yz} + \psi_{zx}\psi_{xyz})\hat{x} + 2(\psi_0\psi_y + \psi_x\psi_{xy} - \psi_z\psi_{yz} + \psi_{zx}\psi_{xyz})\hat{y} + 2(\psi_0\psi_z - \psi_x\psi_{zx} + \psi_y\psi_{yz} + \psi_{xy}\psi_{xyz})\hat{z}$

So, in total we have:

$\psi^*\psi = \rho + \vec{j}$

With:

$\rho = {\psi_0}^2 + {\psi_x}^2 + {\psi_y}^2 + {\psi_z}^2 + {\psi_{yz}}^2 + {\psi_{zx}}^2 + {\psi_{xy}}^2 + {\psi_{xyz}}^2$

And:

$\vec{j} = 2(\psi_0\psi_y + \psi_x\psi_{xy} - \psi_z\psi_{yz} + \psi_{zx}\psi_{xyz})\hat{x} + 2(\psi_0\psi_y + \psi_x\psi_{xy} - \psi_z\psi_{yz} + \psi_{zx}\psi_{xyz})\hat{y} + 2(\psi_0\psi_z - \psi_x\psi_{zx} + \psi_y\psi_{yz} + \psi_{xy}\psi_{xyz})\hat{z}$

Annex A2. Showing that the bra-ket product in Geometric Algebra is equivalent to the operations in Matrix Algebra

For this Annex, we will use the paper, that is very clear on how to operate with matrices in Quantum mechanics. I used the same paper in to make a one-to-one map of Dirac equation between geometric algebra and matrix algebra.

If we consider a general spinor in matrix algebra:

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \begin{pmatrix}\psi_{1r} + i\psi_{1i}\\ \psi_{2r} + i\psi_{2i}\\ \psi_{3r} + i\psi_{3i}\\ \psi_{4r} + i\psi_{4i}\end{pmatrix}$$

In we obtained the following relations to make a mapping between Geometric Algebra and matrix algebra with the coordinate frame (or frame orientation) used in [5]:

$\psi_{1r} = -\psi_y$

$\psi_{1i} = -\psi_x$

$\psi_{2i} = \psi_z$

$\psi_{3r} = -\psi_{yz}$

$\psi_{3i} = \psi_{zx}$

$\psi_{4r} = \psi_{xy}$

Also, we obtained the following two equations, but with an opposite sign. In this case, we have to reverse their sign for the one-to-one map to work. This does not lose any generality as these relations completely free, we are just changing nomenclatures, to be able to compare apples with apples (to have the coordinate system pointing in the same direction in both cases).

$\psi_{2r} = -\psi_{xyz}$

$\psi_{4i} = -\psi_0$

So, applying these relations, we get:

$$\psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \begin{pmatrix}-\psi_y - i\psi_x\\ -\psi_{xyz} + i\psi_z\\ -\psi_{yz} + i\psi_{zx}\\ \psi_{xy} - i\psi_0\end{pmatrix} = \begin{pmatrix}-\psi_y - \psi_x\hat{x}\hat{y}\hat{z}\\ -\psi_{xyz} + \psi_z\hat{x}\hat{y}\hat{z}\\ -\psi_{yz} + \psi_{zx}\hat{x}\hat{y}\hat{z}\\ \psi_{xy} - \psi_0\hat{x}\hat{y}\hat{z}\end{pmatrix}$$

$\psi^\dagger = (\psi^*)^T =$

$(-\psi_y + \psi_x\hat{x}\hat{y}\hat{z} \quad -\psi_{xyz} - \psi_z\hat{x}\hat{y}\hat{z} \quad -\psi_{yz} - \psi_{zx}\hat{x}\hat{y}\hat{z} \quad \psi_{xy} + \psi_0\hat{x}\hat{y}\hat{z})$

We see in that the probability is defined as:

$\rho = \bar{\psi}\,\gamma^0\psi = \psi^\dagger\gamma^0\gamma^0\psi = \psi^\dagger\psi$

This is:

$\rho = \psi^\dagger\psi =$

$$(-\psi_y + \psi_x\hat{x}\hat{y}\hat{z} \quad -\psi_{xyz} - \psi_z\hat{x}\hat{y}\hat{z} \quad -\psi_{yz} - \psi_{zx}\hat{x}\hat{y}\hat{z} \quad \psi_{xy} + \psi_0\hat{x}\hat{y}\hat{z})\begin{pmatrix}-\psi_y - \psi_x\hat{x}\hat{y}\hat{z}\\ -\psi_{xyz} + \psi_z\hat{x}\hat{y}\hat{z}\\ -\psi_{yz} + \psi_{zx}\hat{x}\hat{y}\hat{z}\\ \psi_{xy} - \psi_0\hat{x}\hat{y}\hat{z}\end{pmatrix} = {\psi_y}^2 + {\psi_x}^2 + {\psi_{xyz}}^2 + {\psi_z}^2 + {\psi_{yz}}^2 + {\psi_{zx}}^2 + {\psi_{xy}}^2 + {\psi_0}^2$$

As you can check this value is the same as the one, we have obtained in 29.2 using Geometric Algebra.

Also, following we have that the gamma matrices used in matrix algebra are:

$$\gamma^0 = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

$$\gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma^5 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

If we perform the following products that we will need later, we get:

$$\gamma^0\gamma^1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^0\gamma^2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^0\gamma^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Now, checking we see that:

$j^\mu = \bar{\psi}\gamma^\mu\psi = \psi^\dagger\gamma^0\gamma^\mu\psi$

If we start with the x axis that corresponds with μ=1 we have:

$j^x = j^1 = \bar{\psi}\gamma^1\psi = \psi^\dagger\gamma^0\gamma^1\psi$

We have calculated the product of the γ's before, so operating we have:

$$(-\psi_y + \psi_x\hat{x}\hat{y}\hat{z} \quad -\psi_{xyz} - \psi_z\hat{x}\hat{y}\hat{z} \quad -\psi_{yz} - \psi_{zx}\hat{x}\hat{y}\hat{z} \quad \psi_{xy} + \psi_0\hat{x}\hat{y}\hat{z})$$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -\psi_y - \psi_x\hat{x}\hat{y}\hat{z} \\ -\psi_{xyz} + \psi_z\hat{x}\hat{y}\hat{z} \\ -\psi_{yz} + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_{xy} - \psi_0\hat{x}\hat{y}\hat{z} \end{pmatrix}$$

$$= (-\psi_y + \psi_x\hat{x}\hat{y}\hat{z} \quad -\psi_{xyz} - \psi_z\hat{x}\hat{y}\hat{z} \quad -\psi_{yz} - \psi_{zx}\hat{x}\hat{y}\hat{z} \quad \psi_{xy} + \psi_0\hat{x}\hat{y}\hat{z})$$

$$\begin{pmatrix} \psi_{xy} - \psi_0\hat{x}\hat{y}\hat{z} \\ -\psi_{yz} + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ -\psi_{xyz} + \psi_z\hat{x}\hat{y}\hat{z} \\ -\psi_y - \psi_x\hat{x}\hat{y}\hat{z} \end{pmatrix} = -\psi_y\psi_{xy} + \psi_x\psi_0 + \psi_{xyz}\psi_{yz} + \psi_z\psi_{zx} +$$

$$\psi_{yz}\psi_{xyz} + \psi_{zx}\psi_z - \psi_{xy}\psi_y + \psi_0\psi_x = 2(-\psi_y\psi_{xy} + \psi_x\psi_0 + \psi_{xyz}\psi_{yz} + \psi_z\psi_{zx})$$

We can see that above result is the same as the one obtained in Geometric Algebra.

For the axis y we make the same.

$j^y = j^2 = \bar{\psi}\gamma^2\psi = \psi^\dagger\gamma^0\gamma^1\psi$

$$(-\psi_y + \psi_x\hat{x}\hat{y}\hat{z} \quad -\psi_{xyz} - \psi_z\hat{x}\hat{y}\hat{z} \quad -\psi_{yz} - \psi_{zx}\hat{x}\hat{y}\hat{z} \quad \psi_{xy} + \psi_0\hat{x}\hat{y}\hat{z})$$

$$\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -\psi_y - \psi_x\hat{x}\hat{y}\hat{z} \\ -\psi_{xyz} + \psi_z\hat{x}\hat{y}\hat{z} \\ -\psi_{yz} + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_{xy} - \psi_0\hat{x}\hat{y}\hat{z} \end{pmatrix} =$$

$$(-\psi_y + \psi_x\hat{x}\hat{y}\hat{z} \quad -\psi_{xyz} - \psi_z\hat{x}\hat{y}\hat{z} \quad -\psi_{yz} - \psi_{zx}\hat{x}\hat{y}\hat{z} \quad \psi_{xy} + \psi_0\hat{x}\hat{y}\hat{z})$$

$$\begin{pmatrix} 0 & 0 & 0 & -\hat{x}\hat{y}\hat{z} \\ 0 & 0 & \hat{x}\hat{y}\hat{z} & 0 \\ 0 & -\hat{x}\hat{y}\hat{z} & 0 & 0 \\ \hat{x}\hat{y}\hat{z} & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -\psi_y - \psi_x\hat{x}\hat{y}\hat{z} \\ -\psi_{xyz} + \psi_z\hat{x}\hat{y}\hat{z} \\ -\psi_{yz} + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_{xy} - \psi_0\hat{x}\hat{y}\hat{z} \end{pmatrix} =$$

$$(-\psi_y + \psi_x\hat{x}\hat{y}\hat{z} \quad -\psi_{xyz} - \psi_z\hat{x}\hat{y}\hat{z} \quad -\psi_{yz} - \psi_{zx}\hat{x}\hat{y}\hat{z} \quad \psi_{xy} + \psi_0\hat{x}\hat{y}\hat{z})$$

$$\begin{pmatrix} -\psi_{xy}\hat{x}\hat{y}\hat{z} - \psi_0 \\ -\psi_{yz}\hat{x}\hat{y}\hat{z} - \psi_{zx} \\ +\psi_{xyz}\hat{x}\hat{y}\hat{z} + \psi_z \\ -\psi_y\hat{x}\hat{y}\hat{z} + \psi_x \end{pmatrix}$$

$$= (-\psi_y + \psi_x\hat{x}\hat{y}\hat{z} \quad -\psi_{xyz} - \psi_z\hat{x}\hat{y}\hat{z} \quad -\psi_{yz} - \psi_{zx}\hat{x}\hat{y}\hat{z} \quad \psi_{xy} + \psi_0\hat{x}\hat{y}\hat{z})$$

$$\begin{pmatrix} -\psi_0 - \psi_{xy}\hat{x}\hat{y}\hat{z} \\ -\psi_{zx} - \psi_{yz}\hat{x}\hat{y}\hat{z} \\ \psi_z + \psi_{xyz}\hat{x}\hat{y}\hat{z} \\ \psi_x - \psi_y\hat{x}\hat{y}\hat{z} \end{pmatrix} = \psi_y\psi_0 + \psi_x\psi_{xy} + \psi_{xyz}\psi_{zx} - \psi_z\psi_{yz} - \psi_{yz}\psi_z +$$

$$\psi_{zx}\psi_{xyz} + \psi_{xy}\psi_x + \psi_0\psi_y = 2(\psi_y\psi_0 + \psi_x\psi_{xy} + \psi_{xyz}\psi_{zx} - \psi_z\psi_{yz})$$

Again, with the same result a

Axis z:

$j^z = j^3 = \bar{\psi}\gamma^3\psi = \psi^\dagger\gamma^0\gamma^3\psi$

$$(-\psi_y + \psi_x\hat{x}\hat{y}\hat{z} \quad -\psi_{xyz} - \psi_z\hat{x}\hat{y}\hat{z} \quad -\psi_{yz} - \psi_{zx}\hat{x}\hat{y}\hat{z} \quad \psi_{xy} + \psi_0\hat{x}\hat{y}\hat{z})$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\begin{pmatrix} -\psi_y - \psi_x\hat{x}\hat{y}\hat{z} \\ -\psi_{xyz} + \psi_z\hat{x}\hat{y}\hat{z} \\ -\psi_{yz} + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ \psi_{xy} - \psi_0\hat{x}\hat{y}\hat{z} \end{pmatrix}$$

$$= (-\psi_y + \psi_x\hat{x}\hat{y}\hat{z} \quad -\psi_{xyz} - \psi_z\hat{x}\hat{y}\hat{z} \quad -\psi_{yz} - \psi_{zx}\hat{x}\hat{y}\hat{z} \quad \psi_{xy} + \psi_0\hat{x}\hat{y}\hat{z})$$

$$\begin{pmatrix} -\psi_{yz} + \psi_{zx}\hat{x}\hat{y}\hat{z} \\ -\psi_{xy} + \psi_0\hat{x}\hat{y}\hat{z} \\ -\psi_y - \psi_x\hat{x}\hat{y}\hat{z} \\ \psi_{xyz} - \psi_z\hat{x}\hat{y}\hat{z} \end{pmatrix} = \psi_y\psi_{yz} - \psi_x\psi_{zx} + \psi_{xyz}\psi_{xy} + \psi_z\psi_0 + \psi_{yz}\psi_y -$$

$$\psi_{zx}\psi_x + \psi_{xy}\psi_{xyz} + \psi_0\psi_z = 2(\psi_y\psi_{yz} - \psi_x\psi_{zx} + \psi_{xyz}\psi_{xy} + \psi_z\psi_0)$$

Same result as 29.3 for the z axis.

So, it has been shown that selecting a coherent position of the axes between matrix algebra and geometric algebra we get the same results, as expected.

It is to be remarked that the matrix algebra in this case, clearly is not optimized. We have 4x4 matrices that only use 4 items maximum different than zero. Geometric Algebra is much more compact in this case, avoiding unnecessary redundancies.

Annex A3. Bra-Ket product in non-Euclidean metric (Orthogonal but not orthonormal)

We apply the following relations, when performing the multiplication:

$\hat{x}^2 = \|\hat{x}\|^2 = g_{xx}$

$\hat{y}^2 = \|\hat{y}\|^2 = g_{yy}$

$\hat{z}^2 = \|\hat{z}\|^2 = g_{zz}$

$\hat{x}\hat{y} = -\hat{y}\hat{x}$

$\hat{y}\hat{z} = -\hat{z}\hat{y}$

$\hat{z}\hat{x} = -\hat{x}\hat{z}$

$$\psi^*\psi = (\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{z}\hat{y} + \psi_{zx}\hat{x}\hat{z} + \psi_{xy}\hat{y}\hat{x} + \psi_{xyz}\hat{z}\hat{y}\hat{x})(\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} + \psi_{xyz}\hat{x}\hat{y}\hat{z}) =$$

$$\psi_0^{\,2} + \psi_0\psi_x\hat{x} + \psi_0\psi_y\hat{y} + \psi_0\psi_z\hat{z} + \psi_0\psi_{yz}\hat{y}\hat{z} + \psi_0\psi_{zx}\hat{z}\hat{x} + \psi_0\psi_{xy}\hat{x}\hat{y} + \psi_0\psi_{xyz}\hat{x}\hat{y}\hat{z} +$$

$$\psi_x\psi_0\hat{x} + \psi_x^{\,2}\|\hat{x}\|^2 + \psi_x\psi_y\hat{x}\hat{y} - \psi_x\psi_z\hat{z}\hat{x} + \psi_x\psi_{yz}\hat{x}\hat{y}\hat{z} -$$

$$\psi_x\psi_{zx}\|\hat{x}\|^2\hat{z} + \psi_x\psi_{xy}\|\hat{x}\|^2\hat{y} + \psi_x\psi_{xyz}\|\hat{x}\|^2\hat{y}\hat{z} +$$

$$\psi_y\psi_0\hat{y} - \psi_y\psi_x\hat{x}\hat{y} + \psi_y^{\,2}\|\hat{y}\|^2 + \psi_y\psi_z\hat{y}\hat{z} + \psi_y\psi_{yz}\|\hat{y}\|^2\hat{z} +$$

$$\psi_y\psi_{zx}\hat{x}\hat{y}\hat{z} - \psi_y\psi_{xy}\|\hat{y}\|^2\hat{x} + \psi_y\psi_{xyz}\|\hat{y}\|^2 z\hat{x} +$$

$$\psi_y\psi_0\hat{y} - \psi_y\psi_x\hat{x}\hat{y} + \psi_y^{\,2}\|\hat{y}\|^2 + \psi_y\psi_z\hat{y}\hat{z} + \psi_y\psi_{yz}\|\hat{y}\|^2\hat{z} +$$

$$\psi_y\psi_{zx}\hat{x}\hat{y}\hat{z} - \psi_y\psi_{xy}\|\hat{y}\|^2\hat{x} + \psi_y\psi_{xyz}\|\hat{y}\|^2 z\hat{x} +$$

$-\psi_{yz}\psi_0\hat{y}\hat{z} - \psi_{yz}\psi_x\hat{x}\hat{y}\hat{z} + \psi_{yz}\psi_y\|\hat{y}\|^2\hat{z} - \psi_{yz}\psi_z\|\hat{z}\|^2\hat{y} +$
${\psi_{yz}}^2\|\hat{y}\|^2\|\hat{z}\|^2 + \psi_{yz}\psi_{zx}\|\hat{z}\|^2\hat{x}\hat{y} - \psi_{yz}\psi_{xy}\|\hat{y}\|^2\hat{z}\hat{x} +$
$\psi_{yz}\psi_{xyz}\|\hat{y}\|^2\|\hat{z}\|^2\hat{x} -$
$-\psi_{xy}\psi_0\hat{x}\hat{y} + \psi_{xy}\psi_x\|\hat{x}\|^2\hat{y} - \psi_{xy}\psi_y\|\hat{y}\|^2\hat{x} - \psi_{xy}\psi_z\hat{x}\hat{y}\hat{z} +$
$\psi_{xy}\psi_{yz}\|\hat{y}\|^2\hat{z}\hat{x} - \psi_{xy}\psi_{zx}\|\hat{x}\|^2\hat{y}\hat{z} + {\psi_{xy}}^2\|\hat{x}\|^2\|\hat{y}\|^2 +$
$\psi_{xy}\psi_{xyz}\|\hat{x}\|^2\|\hat{y}\|^2\hat{z} -$
$-\psi_{xyz}\psi_0\hat{x}\hat{y}\hat{z} - \psi_{xyz}\psi_x\|\hat{x}\|^2\hat{y}\hat{z} - \psi_{xyz}\psi_y\|\hat{y}\|^2\hat{z}\hat{x} - \psi_{xyz}\psi_z\|\hat{z}\|^2\hat{x}\hat{y} +$
$\psi_{xyz}\psi_{yz}\|\hat{y}\|^2\|\hat{z}\|^2\hat{x} + \psi_{xyz}\psi_{zx}\|\hat{x}\|^2\|\hat{z}\|^2\hat{y} + \psi_{xyz}\psi_{xy}\|\hat{x}\|^2\|\hat{y}\|^2\hat{z} +$
${\psi_{xyz}}^2\|\hat{x}\|^2\|\hat{y}\|^2\|\hat{z}\|^2$

The scalar part is:
$\rho = {\psi_0}^2 + {\psi_x}^2\|\hat{x}\|^2 + {\psi_y}^2\|\hat{y}\|^2 + {\psi_z}^2\|\hat{z}\|^2 + {\psi_{yz}}^2\|\hat{y}\|^2\|\hat{z}\|^2 +$
${\psi_{zx}}^2\|\hat{z}\|^2\|\hat{x}\|^2 + {\psi_{xy}}^2\|\hat{x}\|^2\|\hat{y}\|^2 + {\psi_{xyz}}^2\|\hat{x}\|^2\|\hat{y}\|^2\|\hat{z}\|^2$
$\rho = {\psi_0}^2 + {\psi_x}^2 g_{xx} + {\psi_y}^2 g_{yy} + {\psi_z}^2 g_{zz} + {\psi_{yz}}^2 g_{yy}g_{zz} + {\psi_{zx}}^2 g_{zz}g_{xx} +$
${\psi_{xy}}^2 g_{xx}g_{yy} + {\psi_{xyz}}^2 g_{xx}g_{yy}g_{zz}$

In x axis:
$\psi_0\psi_x + \psi_x\psi_0 - \psi_y\psi_{xy}\|\hat{y}\|^2 + \psi_z\psi_{zx}\|\hat{z}\|^2 + \psi_{yz}\psi_{xyz}\|\hat{y}\|^2\|\hat{z}\|^2 +$
$\psi_{zx}\psi_z\|\hat{z}\|^2 - \psi_{xy}\psi_y\|\hat{y}\|^2 + \psi_{xyz}\psi_{yz}\|\hat{y}\|^2\|\hat{z}\|^2$
$2(\psi_0\psi_x - \psi_y\psi_{xy}\|\hat{y}\|^2 + \psi_z\psi_{zx}\|\hat{z}\|^2 + \psi_{yz}\psi_{xyz}\|\hat{y}\|^2\|\hat{z}\|^2)$
$\psi_0\psi_x + \psi_x\psi_0 - \psi_y\psi_{xy}g_{yy} + \psi_z\psi_{zx}g_{zz} + \psi_{yz}\psi_{xyz}g_{yy}g_{zz} +$
$\psi_{zx}\psi_z g_{zz} - \psi_{xy}\psi_y g_{yy} + \psi_{xyz}\psi_{yz}g_{yy}g_{zz}$
$2(\psi_0\psi_x - \psi_y\psi_{xy}g_{yy} + \psi_z\psi_{zx}g_{zz} + \psi_{yz}\psi_{xyz}g_{yy}g_{zz})$

In y axis:
$+\psi_0\psi_y + \psi_x\psi_{xy}\|\hat{x}\|^2 + \psi_y\psi_0 - \psi_z\psi_{yz}\|\hat{z}\|^2 - \psi_{yz}\psi_z\|\hat{z}\|^2 +$
$\psi_{zx}\psi_{xyz}\|\hat{z}\|^2\|\hat{x}\|^2 + \psi_{xy}\psi_x\|\hat{x}\|^2 + \psi_{xyz}\psi_{zx}\|\hat{x}\|^2\|\hat{z}\|^2$
$2(+\psi_0\psi_y + \psi_x\psi_{xy}\|\hat{x}\|^2 - \psi_z\psi_{yz}\|\hat{z}\|^2 + \psi_{zx}\psi_{xyz}\|\hat{z}\|^2\|\hat{x}\|^2)$
$2(+\psi_0\psi_y + \psi_x\psi_{xy}\|\hat{x}\|^2 - \psi_z\psi_{yz}\|\hat{z}\|^2 + \psi_{zx}\psi_{xyz}\|\hat{z}\|^2\|\hat{x}\|^2)$
$2(+\psi_0\psi_y + \psi_x\psi_{xy}g_{xx} - \psi_{yz}\psi_z g_{zz} + \psi_{zx}\psi_{xyz}g_{zz}g_{xx})$

In z axis:
$+\psi_0\psi_z - \psi_x\psi_{zx}\|\hat{x}\|^2 + \psi_y\psi_{yz}\|\hat{y}\|^2 + \psi_z\psi_0 + \psi_{yz}\psi_y\|\hat{y}\|^2 -$
$\psi_{zx}\psi_x\|\hat{x}\|^2 + \psi_{xy}\psi_{xyz}\|\hat{x}\|^2\|\hat{y}\|^2 + \psi_{xyz}\psi_{xy}\|\hat{x}\|^2\|\hat{y}\|^2$
$2(+\psi_0\psi_z - \psi_x\psi_{zx}\|\hat{x}\|^2 + \psi_y\psi_{yz}\|\hat{y}\|^2 + \psi_{xy}\psi_{xyz}\|\hat{x}\|^2\|\hat{y}\|^2)$
$+\psi_0\psi_z - \psi_x\psi_{zx}g_{xx} + \psi_y\psi_{yz}g_{yy} + \psi_z\psi_0 + \psi_{yz}\psi_y g_{yy} - \psi_{zx}\psi_x g_{xx} +$
$\psi_{xy}\psi_{xyz}g_{xx}g_{yy} + \psi_{xyz}\psi_{xy}g_{xx}g_{yy}$
$2(+\psi_0\psi_z - \psi_x\psi_{zx}g_{xx} + \psi_y\psi_{yz}g_{yy} + \psi_{xy}\psi_{xyz}g_{xx}g_{yy})$

In yz plane:
$+\psi_0\psi_{yz} + \psi_x\psi_{xyz}\|\hat{x}\|^2 + \psi_y\psi_z - \psi_z\psi_y - \psi_{yz}\psi_0 + \psi_{zx}\psi_{xy}\|\hat{x}\|^2 -$
$\psi_{xy}\psi_{zx}\|\hat{x}\|^2 - \psi_{xyz}\psi_x\|\hat{x}\|^2 = 0$

In zx plane:
$+\psi_0\psi_{zx} - \psi_x\psi_z + \psi_y\psi_{xyz}\|\hat{y}\|^2 + \psi_z\psi_x\hat{z}\hat{x} - \psi_{yz}\psi_{xy}\|\hat{y}\|^2 - \psi_{zx}\psi_0 +$
$\psi_{xy}\psi_{yz}\|\hat{y}\|^2 - \psi_{xyz}\psi_y\|\hat{y}\|^2 = 0$

In xy plane:
$+\psi_0\psi_{xy} + \psi_x\psi_y - \psi_y\psi_x + \psi_z\psi_{xyz}\|\hat{z}\|^2 + \psi_{yz}\psi_{zx}\|\hat{z}\|^2 -$
$\psi_{zx}\psi_{yz}\|\hat{z}\|^2 - \psi_{xy}\psi_0 - \psi_{xyz}\psi_z\|\hat{z}\|^2 = 0$

In xyz plane:
$+\psi_0\psi_{xyz} + \psi_x\psi_{yz} + \psi_y\psi_{zx} + \psi_z\psi_{xy} - \psi_{yz}\psi_x - \psi_{zx}\psi_y - \psi_{xy}\psi_z -$
$\psi_{xyz}\psi_0 = 0$

Again, we only have the probability (scalars) and the fermionic current (vectors). The planes and the trivector sum zero.

So, summing up:
$\psi^\dagger\psi = \psi^*\psi = \rho + \vec{j}$
Being for this case (orthogonal but non-orthonormal, non-Euclidean case):
$\rho = {\psi_0}^2 + {\psi_x}^2 g_{xx} + {\psi_y}^2 g_{yy} + {\psi_z}^2 g_{zz} + {\psi_{yz}}^2 g_{yy}g_{zz} + {\psi_{zx}}^2 g_{zz}g_{xx} +$
${\psi_{xy}}^2 g_{xx}g_{yy} + {\psi_{xyz}}^2 g_{xx}g_{yy}g_{zz}$
And:
$\vec{j} = 2(\psi_0\psi_x - \psi_y\psi_{xy}g_{yy} + \psi_z\psi_{zx}g_{zz} + \psi_{yz}\psi_{xyz}g_{yy}g_{zz})\hat{x} +$
$2(+\psi_0\psi_y + \psi_x\psi_{xy}g_{xx} - \psi_{yz}\psi_z g_{zz} + \psi_{zx}\psi_{xyz}g_{zz}g_{xx})\hat{y} + 2(+\psi_0\psi_z -$
$\psi_x\psi_{zx}g_{xx} + \psi_y\psi_{yz}g_{yy} + \psi_{xy}\psi_{xyz}g_{xx}g_{yy})\hat{z}$

Annex A4. Bra-Ket product in non-Euclidean metric (Non orthogonal and not orthonormal)

We apply the following relations:
$\hat{x}^2 = \|\hat{x}\|^2 = g_{xx}$
$\hat{y}^2 = \|\hat{y}\|^2 = g_{yy}$
$\hat{z}^2 = \|\hat{z}\|^2 = g_{zz}$
$\hat{x}\hat{y} = 2g_{xy} - \hat{y}\hat{x}$
$\hat{y}\hat{z} = 2g_{yz} - \hat{z}\hat{y}$
$\hat{z}\hat{x} = 2g_{zx} - \hat{x}\hat{z}$
$\psi^*\psi = (\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{z}\hat{y} + \psi_{zx}\hat{x}\hat{z} + \psi_{xy}\hat{y}\hat{x} +$
$\psi_{xyz}\hat{z}\hat{y}\hat{x})(\psi_0 + \psi_x\hat{x} + \psi_y\hat{y} + \psi_z\hat{z} + \psi_{yz}\hat{y}\hat{z} + \psi_{zx}\hat{z}\hat{x} + \psi_{xy}\hat{x}\hat{y} +$
$\psi_{xyz}\hat{x}\hat{y}\hat{z}) =$
${\psi_0}^2 + \psi_0\psi_x\hat{x} + \psi_0\psi_y\hat{y} + \psi_0\psi_z\hat{z} + \psi_0\psi_{yz}\hat{y}\hat{z} + \psi_0\psi_{zx}\hat{z}\hat{x} +$
$\psi_0\psi_{xy}\hat{x}\hat{y} + \psi_0\psi_{xyz}\hat{x}\hat{y}\hat{z} +$
$\psi_x\psi_0\hat{x} + {\psi_x}^2\|\hat{x}\|^2 + \psi_x\psi_y\hat{x}\hat{y} + \psi_x\psi_z\hat{x}\hat{z} + \psi_x\psi_{yz}\hat{x}\hat{y}\hat{z} + \psi_x\psi_{zx}\hat{x}\hat{z}\hat{x} +$
$\psi_x\psi_{xy}\|\hat{x}\|^2\hat{y} + \|\hat{x}\|^2\psi_x\psi_{xyz}\hat{y}\hat{z} +$
$\psi_y\psi_0\hat{y} + \psi_y\psi_x\hat{y}\hat{x} + {\psi_y}^2\|\hat{y}\|^2 + \psi_y\psi_z\hat{y}\hat{z} + \|\hat{y}\|^2\psi_y\psi_{yz}\hat{z} +$
$\psi_y\psi_{zx}\hat{y}\hat{z}\hat{x} + \psi_y\psi_{xy}\hat{y}\hat{x}\hat{y} + \psi_y\psi_{xyz}\hat{y}\hat{x}\hat{y}\hat{z} +$
$\psi_z\psi_0\hat{z} + \psi_z\psi_x\hat{z}\hat{x} + \psi_z\psi_y\hat{z}\hat{y} + {\psi_z}^2\|\hat{z}\|^2 + \psi_z\psi_{yz}\hat{z}\hat{y}\hat{z} + \psi_z\psi_{zx}\|\hat{z}\|^2\hat{x} +$
$\psi_z\psi_{xy}\hat{z}\hat{x}\hat{y} + \psi_z\psi_{xyz}\hat{z}\hat{x}\hat{y}\hat{z}$
$+\psi_{yz}\psi_0\hat{z}\hat{y} + \psi_{yz}\psi_x\hat{z}\hat{y}\hat{x} + \psi_{yz}\psi_y\hat{z}\|\hat{y}\|^2 + \psi_{yz}\psi_z\hat{z}\hat{y}\hat{z} +$
${\psi_{yz}}^2\|\hat{z}\|^2\|\hat{y}\|^2 + \psi_{yz}\psi_{zx}\hat{z}\hat{y}\hat{z}\hat{x} + \psi_{yz}\psi_{xy}\hat{z}\hat{y}\hat{x}\hat{y} + \psi_{yz}\psi_{xyz}\hat{z}\hat{y}\hat{x}\hat{y}\hat{z}$
$+\psi_{zx}\psi_0\hat{x}\hat{z} + \psi_{zx}\psi_x\hat{x}\|\hat{z}\|^2 + \psi_{zx}\psi_y\hat{x}\hat{z}\hat{y} + \psi_{zx}\psi_z\hat{x}\hat{z}\hat{x} - \psi_{zx}\psi_{yz}\hat{x}\hat{z}\hat{y}\hat{z} +$
${\psi_{zx}}^2 + \psi_{zx}\psi_{xy}\hat{x}\hat{z}\hat{x}\hat{y} + \psi_{zx}\psi_{xyz}\hat{x}\hat{z}\hat{x}\hat{y}\hat{z} +$
$+\psi_{xy}\psi_0\hat{y}\hat{x} + \psi_{xy}\psi_x\hat{y}\|\hat{x}\|^2 + \psi_{xy}\psi_y\hat{y}\hat{x}\hat{y} + \psi_{xy}\psi_z\hat{y}\hat{x}\hat{z} +$
$\psi_{xy}\psi_{yz}\hat{y}\hat{x}\hat{y}\hat{z} + \psi_{xy}\psi_{zx}\hat{y}\hat{x}\hat{z}\hat{x} + {\psi_{xy}}^2\|\hat{y}\|^2\|\hat{x}\|^2 +$
$\psi_{xy}\psi_{xyz}\|\hat{x}\|^2\|\hat{y}\|^2\hat{z} +$
$+\psi_{xyz}\psi_0\hat{z}\hat{y}\hat{x} + \psi_{xyz}\psi_x\|\hat{x}\|^2\hat{z}\hat{y} + \psi_{xyz}\psi_y\hat{z}\hat{y}\hat{x}\hat{y} + \psi_{xyz}\psi_z\hat{z}\hat{y}\hat{x}\hat{z} +$
$\psi_{xyz}\psi_{yz}\hat{z}\hat{y}\hat{x}\hat{y}\hat{z} + \psi_{xyz}\psi_{zx}\hat{z}\hat{y}\hat{x}\hat{z}\hat{x} + \psi_{xyz}\psi_{xy}\|\hat{x}\|^2\|\hat{y}\|^2\hat{z} +$
${\psi_{xyz}}^2\|\hat{x}\|^2\|\hat{y}\|^2\|\hat{z}\|^2$

And then for the non-simplified products above, you can use the following relations:
$\hat{x}\hat{z} = 2g_{zx} - \hat{z}\hat{x}$
$\hat{x}\hat{z}\hat{x} = (2g_{zx} - \hat{z}\hat{x})\hat{x} = 2g_{zx}\hat{x} - \hat{z}\|\hat{x}\|^2$
$\hat{y}\hat{x} = (2g_{xy} - \hat{x}\hat{y})$
$\hat{y}\hat{z}\hat{x} = \hat{y}(2g_{zx} - \hat{x}\hat{z}) = 2g_{zx}\hat{y} - \hat{y}\hat{x}\hat{z} = 2g_{zx}\hat{y} - (2g_{xy} - \hat{x}\hat{y})\hat{z} =$
$2g_{zx}\hat{y} - 2g_{xy}\hat{z} + \hat{x}\hat{y}\hat{z}$
$\hat{y}\hat{x}\hat{y} = (2g_{xy} - \hat{x}\hat{y})\hat{y} = 2g_{xy}\hat{y} - \hat{x}\|\hat{y}\|^2$
$\hat{y}\hat{x}\hat{y}\hat{z} = (2g_{xy}\hat{y} - \hat{x}\|\hat{y}\|^2)\hat{z} = 2g_{xy}\hat{y}\hat{z} - \hat{x}\hat{z}\|\hat{y}\|^2$
$\hat{z}\hat{y} = 2g_{yz} - \hat{y}\hat{z}$
$\hat{z}\hat{y}\hat{z} = (2g_{yz} - \hat{y}\hat{z})\hat{z} = 2g_{yz}\hat{z} - \hat{y}\|\hat{z}\|^2$
$\hat{z}\hat{x}\hat{y} = (2g_{zx} - \hat{x}\hat{z})\hat{y} = 2g_{zx}\hat{y} - \hat{x}\hat{z}\hat{y} = 2g_{zx}\hat{y} - \hat{x}(2g_{yz} - \hat{y}\hat{z}) =$
$2g_{zx}\hat{y} - 2g_{yz}\hat{x} + \hat{x}\hat{y}\hat{z}$
$\hat{z}\hat{x}\hat{y}\hat{z} = (2g_{zx} - \hat{x}\hat{z})\hat{y}\hat{z} = 2g_{zx}\hat{y}\hat{z} - \hat{x}\hat{z}\hat{y}\hat{z} = 2g_{zx}\hat{y}\hat{z} - \hat{x}(2g_{yz} -$
$\hat{y}\hat{z})\hat{z} = 2g_{zx}\hat{y}\hat{z} - 2g_{yz}\hat{x} + \hat{x}\hat{y}\|\hat{z}\|^2$
$\hat{z}\hat{y} = 2g_{yz} - \hat{y}\hat{z}$
$\hat{z}\hat{y}\hat{x} = (2g_{yz} - \hat{y}\hat{z})\hat{x} = 2g_{yz}\hat{x} - \hat{y}(2g_{zx} - \hat{x}\hat{z}) = 2g_{yz}\hat{x} - 2g_{zx}\hat{y} +$
$(2g_{xy} - \hat{x}\hat{y})\hat{z} = 2g_{yz}\hat{x} - 2g_{zx}\hat{y} + 2g_{xy}\hat{z} - \hat{x}\hat{y}\hat{z}$
$\hat{z}\hat{y}\hat{z} = (2g_{yz} - \hat{y}\hat{z})\hat{z} = 2g_{yz}\hat{z} - \hat{y}\|\hat{z}\|^2$
$\hat{z}\hat{y}\hat{z}\hat{x} = (2g_{yz} - \hat{y}\hat{z})\hat{z}\hat{x} = 2g_{yz}\hat{z}\hat{x} - \|\hat{z}\|^2\hat{y}\hat{x} = 2g_{yz}\hat{z}\hat{x} -$
$\|\hat{z}\|^2(2g_{xy} - \hat{x}\hat{y}) = 2g_{yz}\hat{z}\hat{x} - 2\|\hat{z}\|^2 2g_{xy} + \|\hat{z}\|^2\hat{x}\hat{y}$
$\hat{z}\hat{y}\hat{x}\hat{y} = \hat{z}(2g_{xy}\hat{y} - \hat{x}\|\hat{y}\|^2) = 2g_{xy}\hat{z}\hat{y} - \|\hat{y}\|^2\hat{z}\hat{x} = 2g_{xy}(2g_{yz} -$
$\hat{y}\hat{z}) - \|\hat{y}\|^2\hat{z}\hat{x} = 4g_{xy}g_{yz} - 2g_{xy}\hat{y}\hat{z} - \|\hat{y}\|^2\hat{z}\hat{x}$
$\hat{z}\hat{y}\hat{x}\hat{y}\hat{z} = (4g_{xy}g_{yz} - 2g_{xy}\hat{y}\hat{z} - \|\hat{y}\|^2\hat{z}\hat{x})\hat{z} = 4g_{xy}g_{yz}\hat{z} -$

$2g_{xy}\|\hat{z}\|^2\hat{y} - \|\hat{y}\|^2(2g_{zx}\hat{x} - \hat{z}\|\hat{x}\|^2) = 4g_{xy}g_{yz}\hat{z} - 2g_{xy}\|\hat{z}\|^2\hat{y} - 2g_{zx}\|\hat{y}\|^2\hat{x} + \|\hat{y}\|^2\|\hat{x}\|^2\hat{z}$

$\hat{x}\hat{z} = 2g_{zx} - \hat{z}\hat{x}$

$\hat{x}\hat{z}\hat{y} = \hat{x}(2g_{yz} - \hat{y}\hat{z}) = 2g_{yz}\hat{x} - \hat{x}\hat{y}\hat{z}$

$\hat{x}\hat{z}\hat{x} = 2g_{zx}\hat{x} - \hat{z}\|\hat{x}\|^2$

$\hat{x}\hat{z}\hat{y}\hat{z} = \hat{x}(2g_{yz}\hat{z} - \hat{y}\|\hat{z}\|^2) = 2g_{yz}\hat{x}\hat{z} - \hat{x}\hat{y}\|\hat{z}\|^2 = 2g_{yz}(2g_{zx} - \hat{z}\hat{x}) - \hat{x}\hat{y}\|\hat{z}\|^2 = 4g_{yz}g_{zx} - 2g_{yz}\hat{z}\hat{x} - \hat{x}\hat{y}\|\hat{z}\|^2$

$\hat{x}\hat{z}\hat{x}\hat{y} = \hat{x}(2g_{zx} - \hat{x}\hat{z})\hat{y} = 2g_{zx}\hat{x}\hat{y} - \|\hat{x}\|^2\hat{z}\hat{y} = 2g_{zx}\hat{x}\hat{y} - \|\hat{x}\|^2(2g_{yz} - \hat{y}\hat{z}) = 2g_{zx}\hat{x}\hat{y} - 2g_{yz}\|\hat{x}\|^2 + \|\hat{x}\|^2\hat{y}\hat{z}$

$\hat{x}\hat{z}\hat{x}\hat{y}\hat{z} = \hat{x}(2g_{zx} - \hat{x}\hat{z})\hat{y}\hat{z} = 2g_{zx}\hat{x} - \|\hat{x}\|^2(2g_{yz}\hat{z} - \hat{y}\|\hat{z}\|^2) = 2g_{zx}\hat{x} - 2g_{yz}\|\hat{x}\|^2\hat{z} - \|\hat{z}\|^2\|\hat{x}\|^2\hat{y}$

$\hat{y}\hat{x}\hat{y} = 2g_{xy}\hat{y} - \hat{x}\|\hat{y}\|^2$

$\hat{y}\hat{x}\hat{z} = (2g_{xy} - \hat{x}\hat{y})\hat{z} = 2g_{xy}\hat{z} - \hat{x}\hat{y}\hat{z}$

$\hat{y}\hat{x}\hat{y}\hat{z} = \hat{y}(2g_{xy} - \hat{y}\hat{x})\hat{z} = 2g_{xy}\hat{y}\hat{z} - \|\hat{y}\|^2(2g_{zx} - \hat{z}\hat{x}) = 2g_{xy}\hat{y}\hat{z} - 2g_{zx}\|\hat{y}\|^2 + \|\hat{y}\|^2\hat{z}\hat{x}$

$\hat{y}\hat{x}\hat{z}\hat{x} = \hat{y}\hat{x}(2g_{zx} - \hat{x}\hat{z}) = 2g_{zx}\hat{y}\hat{x} - \|\hat{x}\|^2\hat{y}\hat{z} = 2g_{zx}(2g_{xy} - \hat{x}\hat{y}) - \|\hat{x}\|^2\hat{y}\hat{z} = 4g_{zx}g_{xy} - 2g_{zx}\hat{x}\hat{y}$

$\hat{z}\hat{y}\hat{x} = 2g_{yz}\hat{x} - 2g_{zx}\hat{y} + 2g_{xy}\hat{z} - \hat{x}\hat{y}\hat{z}$

$\|\hat{x}\|^2\hat{z}\hat{y} = \|\hat{x}\|^2(2g_{yz} - \hat{y}\hat{z}) = 2g_{yz}\|\hat{x}\|^2 - \|\hat{x}\|^2\hat{y}\hat{z}$

$\hat{z}\hat{y}\hat{x}\hat{y} = 4g_{xy}g_{yz} - 2g_{xy}\hat{y}\hat{z} - \|\hat{y}\|^2\hat{z}\hat{x}$

$\hat{z}\hat{y}\hat{x}\hat{z} = (2g_{yz}\hat{x} - 2g_{zx}\hat{y} + 2g_{xy}\hat{z} - \hat{x}\hat{y}\hat{z})\hat{z} = 2g_{yz}\hat{x}\hat{z} - 2g_{zx}\hat{y}\hat{z} + 2g_{xy}\|\hat{z}\|^2 - \|\hat{z}\|^2\hat{x}\hat{y}$

$\hat{z}\hat{y}\hat{x}\hat{y}\hat{z} = 4g_{xy}g_{yz}\hat{z} - 2g_{xy}\|\hat{z}\|^2\hat{y} - 2g_{zx}\|\hat{y}\|^2\hat{x} + \|\hat{y}\|^2\|\hat{x}\|^2\hat{z}$

$\hat{z}\hat{y}\hat{x}\hat{z}\hat{x} = \hat{z}\hat{y}\hat{x}(2g_{zx} - \hat{x}\hat{z}) = 2g_{zx}\hat{z}\hat{y}\hat{x} - \|\hat{x}\|^2\hat{z}\hat{y}\hat{z} = 2g_{zx}(2g_{yz}\hat{x} - 2g_{zx}\hat{y} + 2g_{xy}\hat{z} - \hat{x}\hat{y}\hat{z}) - \|\hat{x}\|^2(2g_{yz}\hat{z} - \hat{y}\|\hat{z}\|^2) = 4g_{zx}g_{yz}\hat{x} - 4g_{zx}g_{zx}\hat{y} + 4g_{xy}\hat{z} - 2g_{zx}\hat{x}\hat{y}\hat{z} - 2\|\hat{x}\|^2g_{yz}\hat{z} + \|\hat{x}\|^2\|\hat{z}\|^2\hat{y}$

As you can imagine, this is a complete nightmare. It is better to make the operation once we know exactly which metric to use and no to try to go the most general case. Anyhow, with the use of a computer with a Geometric Algebra program, for sure, we can get the final result.