ORIGINAL ARTICLE

Involutive Hom-Lie triple systems

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ABSTRACT

In this work we we prove that all involutive Hom-Lie triple systems are whether simple or semi-simple. Moreover, we prove that an involutive simple Lie triple system give a rise of InvolutiveHom-Lie triple system.

Key Words: *Jordan triple system; Lie triple system; Casimir operator; Quadratic lie algebra; TKK construction*

The classification of semisimple Lie algebras with involutions can be found in (1). The Hom-Lie algebras were initially introduced by Hartwig, Larson and Silvestrov in (2) motivated initially by examples of deformed Lie algebras coming from twisted discretizations of vector fields. The Killing form K of g is nondegenerate and $\hat{I}\ddot{y}$ is symmetric with respect to K. In (3), the author studied Hom-Lie triple system using the double extension and gives an inductive description of quadratic Hom-Lie triple system. In this work we recall the definition of involutive Hom-Lie triple systems and some related structure and we prove that all involutive Hom-Lie triple systems are whether simple or semi-simple. Moreover,we prove that an involutive simple Lie triple system give a rise of Involutive Hom-Lie triple system.

Definition 0.1

A Hom-Lie triple system is a triple $(L,[-,-,-],\alpha)$ consisting of a linear space L, a trilinear map $[-,-,-]: L\times L\times L \to L$ and a linear map $\alpha: L\to L$ such that

$[x,y,z]=0$ (skewsymmetry)

$[x,y,z]+[y,z,x]+[z,x,y]=0$ (ternary Jacobi identity)

$$[\alpha(u),\alpha(v),[x,y,z]]$$
$$=[[u,v,x],\alpha(y),\alpha(z)]+[\alpha(x),[u,v,y],\alpha(z)]+[\alpha(x),\alpha(y),[u,v,z]],$$

for all $x,y,z,u,v\varepsilon L$. If Moreover α satisfies $\alpha([x,y,z])=[\alpha(x),\alpha(y),\alpha(z)]$ (resp. $\alpha^2=id_L$) for all $x,y,z\varepsilon L$, we say that $(L,[-,-,-],\alpha)$ is a multiplicative (resp. involutive) Hom-Lie triple system.

A Hom-Lie triple system $(L,[-,-,-],\alpha)$ is said to be regular if α is an automorhism of L.

When the twisting map α is equal to the identity map, we recover the usual notion of Lie triple system (4,5). So, Lie triple systems are examples of Hom-Lie triple systems. If we introduce the right multiplication R defined for all $x,y\varepsilon L$ by $R(x,y)(z):=[x,y,z]$, then the conditions above can be written as follow:

$$R(x,y)=-R(y,x),$$

$$R(x,y)z+R(y,z)x+R(z,x)y=0,$$

$$R(\alpha(u),\alpha(v))[x,y,z]$$
$$=[R(u,v)x,\alpha(y),\alpha(z)]+[\alpha(x),R(u,v)y,\alpha(z)]+[\alpha(x),\alpha(y),R(u,v)z].$$

W e can also introduce the middle (resp. left) multiplication operator $M(x,z)y:=[x,y,z](resp. L(y,z)x:=[x,y,z])$ for all $x,y,z\varepsilon L.$ The equations above can be written in operator form respectively as follows:

$$M(x,y)=-L(x,y) \quad [1]$$

$$M(x,y)-M(y,x)=R(x,y) \text{ for all } x,y\varepsilon L. \quad [2]$$

We can write the equation above as one of the equivalent identities of operators:

$$R(\alpha(u),\alpha(v))R(x,y)-R(\alpha(x),\alpha(y))R(u,v)$$
$$=(R(R(u,v)x,\alpha(y))+R(\alpha(x),R(u,v)y))\alpha.$$

$$R(\alpha(u),\alpha(v))M(x,z)-M(\alpha(x),\alpha(z))R(u,v)$$
$$=(M(R(u,v)x,\alpha(z))+M(\alpha(x),R(u,v)z))\alpha.$$

Definition 0.2

Let $(L[-,-,-],\alpha)$ and $(L',[-,-,-]',\alpha')$ be two two Hom-Lie triple systems (6). A linear map $f: L\to L'$ is a morphism of Hom-Lie triple systems if

$f([x,y,z])=[f(x),f(y),f(z)]'$ and $f\circ\alpha=\alpha'\circ f.$

In particular, if f is invertible, then L' and L' are said to be isomorphic.

Definition 0.3

Let $(L,[-,-,-],\alpha$ be a Hom-Lie triple system and I be a subspace of L. We say that I is an ideal of L if $[I,L,L]\subset I$ and $\alpha(I)\subset I.$

Definition 0.4

A Hom-Lie triple system L is said to be simple (resp. semisimple) if it contains no nontrivial ideal $(resp. Rad(L)=\{0\})$.

According to a result in [?], if A is aMalcev algebra, then $(A,[-,-,-])$ is a Lie triple system with triple product

$$[x,y,z]=2(xy)z-(zx)y-(yz)x. \quad [3]$$

Thus, if A is aMalcev algebra and $\alpha: A\to A$ is an algebra morphism, then, $A_\alpha=(A,[-,-,-]_\alpha=\alpha\circ[-,-,-],\alpha$ is a multiplicative Hom-Lie triple system, where $[-,-,-]$ is the triple product in [3].

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Proposition 0.5

Let L be a Lie triple system and α be an automorphism of L. If L is simple, the L is also simple.

Proof: Since L is not abelian, then L_α is also not abelian. Moreover, let I be an ideal of L_α. For all $x, y \varepsilon L$ and a $a \varepsilon I$ we have,

$[a,x,y]_\alpha \varepsilon I.$

That is,

$[\alpha(a),\alpha(x),\alpha(y)]\varepsilon I.$

Consequently, I is an ideal of L because α is an automorphism. Thus, $I=\{0\}$.

Theorem 0.6

Let $(L,[.,.,.],\theta)$ be an involutiveHom-Lie triple system. Then,

$(L,[.,.,.],\theta)$ is simple or semi-simple. Moreover, in the second case L can be wrien as $L := L_\theta = \mathfrak{I} \oplus \theta(\mathfrak{I})$ where $\mathfrak{I}$ is a simple ideal of L. Conversely, If $(L,[.,.,.],\theta)$ is an involutive simple Lie triple system, then $(L,[.,.,.]_\theta,\theta)$ is an involutive Hom-Lie triple system.

Proof: Suppose that L_θ is not simple and put $\mathfrak{I}$ a minimal ideal of L_θ. We get $[L_\theta,L_\theta,\mathfrak{I}]_\theta$ is an ideal of L_θ which is contained on $\mathfrak{I}$. Thus,

$[L_\theta,L_\theta,\mathfrak{I}]_\theta=\{0\}$ or $[L_\theta,L_\theta,\mathfrak{I}]_\theta=\mathfrak{I}.$

Now, firstly, if $[L_\theta,L_\theta,\mathfrak{I}]_\theta=\{0\}$,then $\theta([L_\theta),\theta(L_\theta),\theta(\mathfrak{I})]_\theta=\{0\}$. That is, $[L,L,\theta(\mathfrak{I})]=\{0\}$, because θ is a bijective linear map. which mean that $\theta(\mathfrak{I})\subset Z(L)=\{0\}$. Thus, $[L_\theta,L_\theta,\mathfrak{I}]_\theta=\mathfrak{I}.$ Hence, $[L,L,\theta(\mathfrak{I})]=\mathfrak{I}$. Which implies that $\theta([L,L,\theta(\mathfrak{I})])=[\theta(L),\theta(L),\theta^2(\mathfrak{I})]=\theta(\mathfrak{I}).$ Consequently,

$[L,L,\mathfrak{I}+\theta(\mathfrak{I})]\subset\mathfrak{I}+\theta(\mathfrak{I}).$

Furthermore,

$\theta(\mathfrak{I}+\theta(\mathfrak{I}))=\theta(\mathfrak{I})+\theta^2(\mathfrak{I})=\theta(\mathfrak{I})+\mathfrak{I}.$

Thus $\mathfrak{I}+\theta(\mathfrak{I})$ is an ideal of $(L,[.,.,.],\theta)$. Since $\mathfrak{I}+\theta(\mathfrak{I})\neq\{0\}$, then $L=\mathfrak{I}+\theta(\mathfrak{I}).$

Now, we have to prove that the summation is direct. In fact, since θ is an automorphism of L_θ, then $\theta(\mathfrak{I})$ is an ideal of L_θ. Thus, $\mathfrak{I}\cap\theta(\mathfrak{I})=\mathfrak{I}$ or $\mathfrak{I}\cap\theta(\mathfrak{I})=\{0\}$ because $\mathfrak{I}$ is minimal. Suppose that $\mathfrak{I}\cap\theta(\mathfrak{I})=\mathfrak{I},$ then $\mathfrak{I}=\theta(\mathfrak{I})$ because θ is bijective. On the other hand,

$[L,L,\mathfrak{I}]=\theta([\theta(L),\theta(L),\theta(\mathfrak{I})])=\theta([L,L,\mathfrak{I}]_\theta)\subset\theta(\mathfrak{I})=\mathfrak{I}.$

Thus, $\mathfrak{I}$ is an ideal of $(L,[.,.,.],\theta)$ and $\mathfrak{I}=L$ because $(L,[.,.,.])$ which contradict the fact that $\mathfrak{I}\neq L$ and $\mathfrak{I}\neq\{0\}$. Consequently, $\mathfrak{I}\cap\theta(\mathfrak{I})=\{0\}$ and $L=\mathfrak{I}\oplus\theta(\mathfrak{I}).$

Let us prove that $\mathfrak{I}$ is a simple ideal of $(L_\theta,[.,.,.]_\theta)$. In fact, $L=L_\theta=\mathfrak{I}\oplus\theta(\mathfrak{I}).$ Since θ is an automorphism of L then θ is an automorphism of L_θ.

$[\theta(\mathfrak{I}),L,L]=\theta([\theta(\mathfrak{I}),L,L]=\theta([\mathfrak{I},\theta(L),\theta(L)]=\theta([\mathfrak{I},L,L])\subset\theta(\mathfrak{I}).$

Thus, $\theta(\mathfrak{I})$ is an ideal of L_θ. Furthermore,

$[\mathfrak{I},\mathfrak{I},\mathfrak{I}]_\theta=[\mathfrak{I}\oplus\theta(\mathfrak{I}),\mathfrak{I}\oplus\theta(\mathfrak{I}),\mathfrak{I}]_\theta=[L_\theta,L_\theta,\mathfrak{I}]=\mathfrak{I}.$ Thus, $\mathfrak{I}$ is a simple ideal of L_θ because it is simple with $[\mathfrak{I},\mathfrak{I}]_theta=\mathfrak{I}$. Consequently, L_θ is semi-simple.

Corollary 0.7

Let $(L,[.,.,.])$ be a Lie triple system with involution θ. Such that, $L=\mathfrak{I}\oplus\theta(\mathfrak{I})$ where $\mathfrak{I}$ is a simple ideal of $(L,[.,.,.])$. Then the Hom-Lie triple system $(L_\theta,[.,.,.]_\theta\theta)$ is simple.

Proof: Let $\mathcal{J}$ be an ideal of L_θ such that $\mathcal{J}\neq\{0\}$. We have

$[L,L,\theta(\mathcal{J})]=[\theta(L),\theta(L),\theta(\mathcal{J})]=[L,L,\mathcal{J}]$

because $L=\theta(L)$ and $\mathcal{J}$ is an ideal of L_θ. Moreover,

$[L,L,\mathcal{J}]=\theta([\theta(L),\theta(L),\theta(\mathcal{J})])=\theta([L,L,\mathcal{J}]_theta)\subset\theta(\mathcal{J})=\mathcal{J},$

because $\mathcal{J}$ is stable under θ since it is an ideal of the Hom-Lie triple system of L_θ. Consequently, $\mathcal{J}$ is an ideal of L. Thus, $\mathcal{J}=\mathfrak{I}$ or $\mathcal{J}=\theta(\mathfrak{I})$ or $\mathcal{J}=L.$ Since $\theta(\mathcal{J})\subset\mathcal{J}$, then $\mathcal{J}\neq\mathfrak{I}$ and $\mathcal{J}\neq\theta(\mathfrak{I}).$ Thus, $\mathcal{J}=\neq\mathfrak{I}\oplus\theta(\mathfrak{I})=L.$

Moreover, since $[L,L,L]=L,$ then $[L_\theta,L_\theta,L_\theta]=L_\theta.$ Thus $(L_\theta,[.,.,.]_\theta\theta)$ is a simple Hom-Lie triple system.

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