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Iterative solutions for variational inclusion problems in Banach spaces

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ABSTRACT

Variational inclusion problems have become the apparatus that is generally used to constrain sundry mathematical equations in other to pguarantee the uniqueness and existence of their solutions. The existence of these solutions was earlier studied and proven for uniform Banach Spaces using accretive operators. In this study, we extend the conditions to hold for arbitrary Banach Spaces using uniform accretive operators.

Key Words: *Accretive operators; Banach spaces*

In recent years, variational inequalities have been extende in different directions and areas of studies, using novel and innovative techniques. One of such generalization is variational inclusions. Several problems that occur in engineering, optimization and control situations can be modeled by free boundary problems which leads to variational inequality and variational inclusion problems, Eq. 1 in appropriate spaces.

Definition

(Variational Inclusion Problem). Let $T,F:E\to CB(E)$ be two set-valued mappings, $D(A)\subset E$ an m-accretive mapping, $g:E\to D(A)$ a single valued mapping and N (., .) : $E\times E\to E$ a nonlinear mapping. For any given function $f\in E$ and $\lambda>0$, we consider the following problem. Find q ∈ E, u 2 T(q), v 2 F(q) such that $f\in N(u,v)+\lambda A(g(q))$ (1)

Where E is a real Banach space and E* is its topological dual space. CB (E) is the family of all non-empty convex (closed) and bounded subsets of E.

The duality pairing between E and E*is defined by inner product $(\langle .,.\rangle)$ if E is a Hilbert space and the Hausdorff metric D (.,.) on CB(E) is defined by

$$D(A,B)=max\left\{\sup_{x\in A}d(x,B),\sup_{y\in B}d(A,y)\right\}$$

Given that $A,B\in CB(E)$, the distances d(x,B) or d(A, y) is defined by

$$d(x,B)=\inf_{y\in B}||x-y|| \quad (2)$$

Also, D (T) denotes the domain of T and the normalized duality map is defined by

$$J(x)=\{f^*\in E^*:\langle x,f^*\rangle=\|x\|^2=\|f^*\|^2\},x\in E \quad (3)$$

Variational inequalities were introduced in early nineteen sixties by Hartman and Stampachia [1].

Lemma I.1

(Micheal's Selection Theorem). Let X and Y be two Banach spaces; $T:X\to 2^E$ a lower semicontinuous mapping with nonempty closed convex values. Then T admits a continuous selection i.e. there exists a continuous mapping $h:X\to Y$ such that $h(x)\subset T(x)$ for each $x\subset X$

Lemma I.2

Let E be a uniformly smooth Banach space and $T:E-2^E$ be a lower semi-continuous and m-accretive mapping. Then the following conditions hold;

(a) T admits a continuous and m-accretive selection

(b) If T is also ϕ strongly accretive, then T admits a continuous m-accretive and ϕ strongly accretive selection.

Lemma I.3

(Nadler's Theorem). Let E be a complete metric space, $T:E\to CB(E)$ be a set-valued mapping then for any given $\in>0$ and for any given $x,y\in E,u\in Tx,v\in Ty$, there exists $v\in Ty$ such that

$$D(u,v)\le(1+e)D(Tx,Ty) \quad (4)$$

Algorithm I.4

(Iterative Sequence). For any given $x_0\in E,u_0\subset Tx_0,v_0\in Fx_0$ compute the sequence $\{x_n\}$, $\{u_n\}$, $\{v_n\}$ by the iterative scheme

$$x_n+1\in(1-\alpha_n)x_n+\alpha(f+x_n-N(u_n,vn))-\lambda A(g(x_n))$$

For $u_n\in Tx_n$,

$$\|u_n-u_{n+1}\|\le\left(1+\frac{1}{n+1}\right)D(Tx_n,Tx_{n+1})$$

For $v_n\in F_{xn}$,

$$\|u_n-u_{n+1}\|\le\left(1+\frac{1}{n+1}\right)D(Tx_n,Tx_{n+1}) \quad (5)$$

Eq. 5, is called the Mann iterative sequence, it is a direct consequence of invoking Michael's Selection theorems [2]. Using Nadler's theorem [3], Chang in [4] proved Lemma I.2, thereby establishing the existence of unique solutions to Variational Inclusion problems using accretive operators in uniform Banach Spaces. In this work we present the extension of Chang's work to arbitrary Banach Spaces using uniformly accretive operators based on the Lipschitz property of T and F.

Preliminaries

Definition I.2.

Let A be a set-valued mapping with domain D(A) and range R(A) in E. A is said to be accretive if for all $x,y\in D(A)$ there exists $j(x-y)\in J(x-y)$ such $\langle u-v,j(x-y)\rangle\ge 0$ (6)

Definition I.3.

Let $\phi:[0,+\infty)\to[0,+\infty)$ be a strictly increasing function with $\phi(0)=0$ then the mapping A is strongly accretive if for any $u\in Ax$, and $v\in Ay$

$$\langle u-v,j(x-y),\rangle\ge\phi(\|x-y\|)\|x-y\| \quad (7)$$

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If ϕ (t) = kt, 0 < k < 1, then A is said to be k-strongly accretive, and said to be m-accretive if A is accretive and (I + rA)D(A) = E for all r > 0, where I is the identity mapping.

Theorem I.5

Let E be a uniform smooth Banach space, $T, F : E \to CB(E)$ and $A : D(A) \subset E \to 2^E$ be three set-valued mappings, q : E−D(A) a single valued mapping satisfying the following conditions;

(i) $A \circ g : E \to 2^E$

(ii) $F : E \to CB(E) is \in -Lipschitz\ contionous; \in > 0$

(iii) $F : E \to CB(E) is \in -Lipschitz\ contionous; \in > 0$

(iv)The mapping $x \mapsto N(x, y) is\ \phi$ strongly accretive with respect to the mapping T, and $\phi : [0, +\infty) \to [0, +\infty)$ is a strictly increasing function with $\phi(0) = 0$

(v) The mapping $y \mapsto N(x, y)$ is accretive with respect to the mapping F

Then, for any given $f \in E, \lambda >)$, there exists $q \in E, u \in T(q), v \in F(q)$ which is a solution to Eq. 1.

Theorem I.6

Let E, T, F, A, g, and N be as in Theorem

I.5 and $\{\alpha_n\}$, be a sequence in the closed interval [0, 1] satisfying the following conditions

(a) $\sum_{n=0} \alpha_n = +\infty$ (b) $\sum_{n=0} \alpha_n = +\infty$

If the ranges *R (I −N(T(.), F(.))* and $(A \circ g)$ are bounded, then for any given $u_0 \in T_{x0}$, $u_0 \in T_{x0}$, $v_o \in Fx_o$, the iterative sequences {xn}, {un} and {vn} defined by Algorithm I.4 converges strongly to the solution q, u, z of the set-valued vibrational inclusion problem of Eq. 1

An equivalent form of Theorem I.6 is given as lemma [5].

Lemma I.7

(Moore and Nnoli). Let {an}, {n} and {n} be real sequences such that $(a)\ \alpha_n \in [0,1]\ (b) \lim_{n\to\infty} = 0\ (c) \sum_{n\geq 0} = \infty\ \ (d) \sigma_n = O(\alpha_n)$ Also, let $\psi : [0, \infty) \to [0, \infty)$ be a strictly increasing function with $\psi(0) = 0$ if $a_{n+1}^2 \leq a_n^2 - \delta\psi(a_{n+1}) + \sigma_n$; then, $a_n \to 0$ as $n \to \infty$.

We are not going to reproduce the proofs of these Theorems and Lemmas, it suffices to indicate their implications, conclusions and some of their rudiments as they are used in the course of this paper. For instance, to prove Theorem I.5, one defines the mapping $S : E \mapsto 2^E$ that is expressed by *Sx = N(Tx, Fx),* $x_1, x_2 \in E$ and invoke Morales [6] to establish that S is m-accretive and ϕ~ strongly accretive and hence use the proof of Lemma I.2(b) to conclude that S admits a continuous and –strongly accretive and m-accretive selection $h + \lambda(A \circ g)$. Then, the theorem and proof of Theorem 5.3 in Kobayashi [7] can be used to show that $h + \lambda(A \circ g)$ is m-accretive and ~strongly accretive. Then $h + \lambda(A \circ g)$ can be used to construct a variational inclusion problem that is a subset of Eq. 1 whose solution parses to Eq. 1 by virtue of uniqueness of the element $\sigma_n = \circ(\delta_n)$. The proof of Theorem I.6 is given in [8] and proof of Lemma I.7 is given in [9]. The same assumption that, $\sigma_n = \circ(\delta_n)$ as $n \to \infty$ with $\sigma_n = \circ(\delta_n)$ is made in the proof of both (Theorem I.6 and Lemma I.7) in other to establish that the sequences *{un}* and *{vn}* are Cauchy in other to achieve the results presented in appendix A. In this study we present a cheaper way to achieve the same result for arbitrary Banach spaces [10].

RESULTS

We begin by presenting and proving the following lemmas, which extends the algebraic property of ϕ –strongly accretive operators to uniform accretive operators.

LemmaII.1.

Let E be a real Banach space, $T, F, :\to 2^E$ two set valued mappings and $N(.,.) : E \times E \to E$ a nonlinear mapping satisfying the following conditions;

(i) The mapping $x \mapsto N(x, y)$ is uniformly accretive with respect to the mapping T

(ii) The mapping $y \mapsto N(x, y)$ is accretive with respect to the mapping F

Then the mapping $x_1, x_2 \in E$ defined by *Sx=N(Tx, Fy)* is uniformly accretive.

Proof: For any given $x_1, x_2 \in E$ and for any $w_i \in Tx_i$, I = 1, 2; there exists $w_i \in Tx_i$ and $v_i \in Fx_i$ such that vi =N (wi, vi). By conditions (i) and (ii) and Definition I.2, we have that

$$\langle u_1 - v_2, j(x_1 - x_2)\rangle = \langle N(w_1, v_1) - N(w_2, v_2), j(x_1 - x_2)\rangle$$

$$= \langle N(w_1, v_1) - N(w_2, v_1), j(x_1 - x_2)\rangle$$

$$+ \langle N(w_2, v_1) - N(w_2, v_2), j(x_1 - x_2)\rangle$$

$$\geq \psi(\|x_1 - x_2\|)$$

Which implies that the mapping S=N (T(.), F(.)) is uniformly accretive.

Lemma II.2 Let E be a real Banach space and $T : X \to 2^Y$ be a lower semi-continuous, m-accretive mapping, then the following conditions holds;

(i) T admits a continuous and m-accretive selection

(ii) In addition, if T is also uniformly accretive, then it admits continuous, m-accretive and uniform accretive selections

Proof: The proof of (i) follows from the proofs of

Lemma I.1 and Lemma I.2 (b). For any given $x, y \in E$ and for any $v \in Ty$, $v \in Ty$, we have from the result of Lemma II.1 that

$$\langle u_1 - v_2, j(x_1 - x_2)\rangle \geq \psi(\|x_1 - x_2\|)$$

Letting $u = h(x) \in Tx, v \in h(y)$ we obtain

$\langle h(x) - h(y), j(x - y)\rangle = \langle u - v, j(x - y)\rangle \geq \psi(\|x_1 - x_2\|)$ which implies that h is uniformly accretive.

Now, since T and F are both Lipschitzian, it follows from Eq. 5, that

$$\|u_n - u_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right) D(Tx_n, Tx_{n+1}) \quad \|v_n - v_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right)\|x_n - x_{n+1}\|$$

In the same vein,

$$\|v_n - v_{n+1}\| \leq \left(1 + \frac{1}{n+1}\right)\|x_n - x_{n+1}\|$$

This implies that given any $x_0 \in E$, $u_0 \in Tx_o$, $v_0 \in Fx_o$ the iterative sequences un and vn are cauchy sequences. Therefore, there exists u^*, $v^* \in E$ such that $n \to \infty, n \to \infty$ as $n \to \infty$. By Lemmas II.1 and II.2 and results in results in [6,5,10] we infer that $h + \lambda(A \circ g)$ too is uniform accretive.

Thus, there exist $v \in F(q)$ and $v \in F(q)$ such (8) and establish the results (u*=v*=u) in Appendix A with 'less' continuity restrictions

$$d(u^*,Tq) \le d(u^*,u_n) + \|x_n - q\| \to 0 \quad (9)$$

$$d(u^*,Tq) \le d(u^*,u_n) + \|x_n - q\| \to 0 \quad (10)$$ so that

$$d(u^*,Tq) \le d(u^*,u_n) + \|x_n - q\| \to 0 \quad (11)$$

$$d(v^*,v) \le d(v^*,v_n) + \|x_n - q\| \to 0 \quad (12)$$

Appendix A

Consequence of Proof of Theorem I.6 and and Lemma I.7

Claim A.1. To prove Theorem I.6 and Lemma I.7, the claim is made that [11-13] $\liminf \|x_n - q\| = \sigma > o$ (13)

Proof of Claim: There exists n_0 such that for any $n \ge n_o$,

$$\|x_n - q\| > \frac{\sigma}{2} > 0$$

$$\|x_{n+1} - q\|^2 + \alpha_n \beta_n - 2\alpha_n \phi(\frac{\delta}{2})\frac{\delta}{2}$$

$$= \|x_n - q\|^2 - \alpha_n \phi \delta \frac{\delta}{2}$$

$\exists n_1 > 0$

such that $\forall n_1, \beta < \frac{1}{2}\phi\delta\frac{\delta}{2}$

Let $N_0 = \max\{n_0, n_1\}$ then for all $n \ge N_0$

we have $\|x_{n+1} - q\|^2 \le \|x_n - q\|^2 - \frac{1}{2}\alpha_n \delta \frac{\delta}{2}$

Thus,

$$\frac{1}{2}\alpha_n \phi \delta \frac{\delta}{2} \le \|x_n - q\|^2 - \|x_{n+1} - q\|^2$$

$$\frac{\delta}{2}\phi(\frac{\delta}{2})\sum_{n \ge N_0} \alpha_n \le \|x_{N0} - q\|^2 \le +\infty$$

$\Rightarrow \sum \alpha_n \le \infty \Rightarrow \Leftarrow !$ So, lim inf $\|x_n - q\| = \sigma > 0$ and there exists $x_{nj} \subset x_n$ such that $x_{nj} \to q$ as $j \to \infty$

$x_{n_{j+1}} = (1+\alpha_{nj})\alpha_{nj} + \alpha_{nj}\alpha_{nj} \to q$ as $j \to \infty$. That is, $\forall n \ge 0$ Which implies that $n \to \infty$ as $n \to \infty$. Now using the fact that T *is* μ-Lipschitzian and F is $\in$ Lipschitzian, it follows that from Eq. 5 that

$$\|u_n - u_{n+1}\| \le \left(1 + \frac{1}{n+1}\right) D(T(x_n), T(x_{n+1}))$$

$$\le \left(1 + \frac{1}{n+1}\right) \|x_n - x_{n+1}\|$$

And

$$\|v_n - v_{n+1}\| \le \left(1 + \frac{1}{n+1}\right) D(F(x_n), F(x_{n+1}))$$

$$\le \in \left(1 + \frac{1}{n+1}\right) \|x_n - x_{n+1}\|$$

This result implies that the sequences u_n and v_n are Cauchy sequences. Therefore, there exists u*, $v^* \in E$ such that $u_n \to v^*$, $u_n \to v^*$ as $u^* = v^*$. Next, we prove that $u^* = v^*$ In fact, since,

$$d(u^*,Tq) \le d(u^*,u_n) + d(u_n,Tq) \le d(u^*,u_n) + D(Tx_n,Tq)$$

$$\le d(u^*,u_n) + \mu\|x_n - q\| \to 0 \quad (14)$$

Result of Eq. 14 implies that $u^* \in T(q)$. Similarly, Eq. 15 also implies that [14,15] $v^* \in Fq$.

$$d(v^*,Tq) \le d(v^*,v_n) + d(v_n,Fq)$$

$$\le d(v^*,v_n) + D(Fx_n,Fq)$$

$$\le d(v^*,v_n) + \in \|x_n - q\| \to 0 \quad (15)$$

It remains to show [16] that u*= v*= u. But Eq. 16 and 17 clearly shows this.

$$\le d(u^*,u_n) + D(Tx_n,Tq)$$

$$\le d(u^*,u_n) + D(Tx_n,Tq) \le d(u^*,u_n) + \mu\|x_n - q\| \to 0$$

(16)

Which implies that $u^* = u$ and since;

$$\le d(u^*,u_n) + D(Tx_n,Tq)$$

$$\le d(u^*,u_n) + D(Tx_n,Tq)$$

$$\le d(u^*,u_n) + \in \|x_n - q\| \to 0 \quad (17)$$

This implies u*= v. [17] Summing up the above argument we conclude that the sequences x_n, u_n and v_n defined by Eq. 5, converges strongly to solution (q, u, v) of problem 1 respectively.

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