Linear map and spin I. n-focal tensor and partition function

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ABSTRACT

Hyperelliptic theta functions are set in context to ambiguity of using epipolar geometry on twisted cubic curves. Complex multiplication on elliptic curves with ambiguous correlations is set in context to one-dimensional complex maps. Chaotic continued fractions are set in context to ternary continued fractions and to the elliptic addition theorem, Poncelet closure and scattering amplitudes in quantum statistics.

Key Words: *Continued fractions, addition theorem, Riemann surfaces, elliptic theta functions, twisted cubic*

INTRODUCTION

Dynamical systems $\frac{dx}{dt} = AX$ are exactly integrable for X dimension X ≦ 4 on genus g ≦ 2 curves E_λ for continuous time t ≃ u = (u_0 ... $u_{g\text{-}1}$) where uniformization u and differential d_X describe a Riemann surface Xg of genus g in space $X \in \mathbb{P}^3$ [1,2].

In distinction, det A=0 collinear involutions constitute singular systems. Dynamics of singular systems requires discrete iterated maps $u^{\circ k}$ ($k \in \mathbb{N}$) with dynamical time k.

Weierstrass Sigma and Zeta functions σ(μ, L) and ζ(u, L) as well theta functions ϑ(u, L) correspond to quantum states on X_1 or X_2 [3-5].

The present paper approaches quantum statistics on bifurcating, layered X embedded into projective space P^3, projective plane P^2 and projective line P^1. Spacetime is set in context to hyperelliptic theta on X_2 where the lattice L is a sum $L_0 \oplus L_1$ of tori X_I [6]. The Weierstrass ℘-function ℘ (u, L) projects from torus T=C/(Z+Zω) for a Lattice L of period ω to foliations of a 2 sphere S^2 [7].

The Riemann-Hurwitz formula states that a curve of genus ≥ 2 does not admit rational self- maps of degree ≥ 2. A genus on curve in E_λ in P^1 with legendre parameter λ is isomorphic to a plane curve E_a with Hesse parameter a. Rational self-maps f of x ∈ P^2 leave invariant $E_a \simeq E_\lambda$ [8]. Iterated x are on the entire sphere S^2 (Julia set).

Self- maps f of degree $\deg_x f = 2$ are of particular interest. Here a subset $\gamma_H \subset f$ is investigated where γ_H are rational Hermite substitutions as quadratic maps of a cubic polynomial.

Elliptic curves E_λ can be regarded as μ hyperelliptic curves with variable λ [9]. E_λ depends on uniformisers of modular groups, elliptic units, and modular units [10,11]. Period doubling is defined by multiplication of variable z with modular units (3.6) realizing the Kronecker Weber Hilbert Theorem (KWHT) by generating cyclic fields.

The paper proves equivalence between Poncelet closure, addition theorems of elliptic functions and the existence of a 2-power generator μ for involutions and quaternary continued fractions which result from cycles of quadruples k-1, k-2, k-3 and k-4. In the following a cycle describes a cyclotomic field whereas a period describes an elliptic field being two- periodic. In this sense the results of the present paper contribute to KHWT. A period of a continued fraction corresponds to a complex field $\mathbb{Q}[\sqrt{d}]$.

A Continued Fraction (CF) via unimodular collineations $M(a) = \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} \in \{GL(2,\mathbb{Z}), GL(2, \mathbb{Q}[\sqrt{d}])\}$ is an abelian extension of a rational number field with periods $\{\bar{a}_1\}$ of sequences $\{a_1\}$. One cycle $C_M(1)$ related to a given constant K_1 appears decomposing periods as follows $\{\bar{a}_1\} \to \{\bar{a}_1\}\ \{\bar{a}_1\}$. Ternary continued fractions with unimodular collineations M(a_1, a_2) may exhibit two cycles $C_M(2)$ caused by periods $\{\bar{a}_1\}$, $\{\bar{a}_2\}$ of sequences $\{a_1\}$ and $\{\bar{a}_2\}\ \{\bar{a}_2\}$ are called Bifurcating Continued Fractions (BCF) [12]. Hermite's problem for describing a cubic irrationality ∂ requires a BCF with at least two cycles $C_M(2)$. Periods $\{\bar{a}_i\}$ of sequences $\{a_i\}$ are equivalent to a fraction $\frac{p^{(i)}}{q^{(i)}} \in \mathbb{Q}$

The Weierstrass function ℘μ is invariant with respect to the tent map T_c with c ∈ ℕ or μ_1 of (3.10) [13]. In the simplest case the sequence {ζ} is given by a Cantor string (3.13) which is chaotic. In distinction a

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relation is drawn between periods of sequences $\{a_i\}$ and the existence of a constant K_i of a_i power cyclotomic field. Periods $\{\bar{a}_i\}$ of sequences $\{a_i\}$ are reflected in iterated sequences $\{u^{\circ k}\}, \{\zeta(u^{\circ k})\} and \{\wp(u^{\circ k})\}$.

The continued fraction algorithm describes rational solutions via unimodular collineations. Quaternary continued fractions whose unimodular collineations via $M(a_1, a_2, a_3)$ may exhibit three cycles $C_M(3)$ caused by periods $\{\bar{a}_1\}$,$\{\bar{a}_2\}$ and $\{\bar{a}_3\}$ of sequences $\{a_1\}$,$\{a_2\}$ and $\{a_3\}$ are called Chaotic Continued Fraction (CCF).

The paper relates quaternary continued fraction algorithms for unimodular collineations M (a_1, a_2, a_3) with det M=1 to four SE(3) rigid transformations. Cycles $C_M(2)$ and $C_M(3)$ allow a highly-composite (MNT) as a fast decomposition algorithm.

Three points $\lambda_k = \frac{\wp\left(u^{\circ k}\right)-\wp\left(u^{\circ k+1}\right)}{\wp\left(u^{\circ k+1}\right)-\wp\left(u^{\circ k+2}\right)}$ create elliptic units. The addition theorem (4.1) creates four pseudo random points (1,2,3,4) = ($p_{k\text{-}1}$, $p_{k\text{-}2}$, $p_{k\text{-}3}$, $p_{k\text{-}4}$) of a generalized Weierstrass function differing from $\wp$ by a Hermite transformation $\gamma\wp$.

At $k = 2^{2^i}$ the addition theorem, Poncelet involutions and CCF are isomorph to a SE(3) step. A cycle is equivalent to a cycle of quadruples s=0,1,0',1' or k-1 k-2, k-3, k-4 or i = 1,2,3,4 which corresponds to a Frobenius map $x \rightarrow x^2$

The power tower $2^{\cdot^{\cdot^{2}}}$ creates cycles at the third order $2^{2^{K_i}}$ where K_i is the number of algorithmic steps to catch a cycle $\{\bar{a}i\}$ via involutions $i_x \rightarrow i_x \circ i_x$, $i_u \rightarrow i_u \circ i_u = \alpha \circ \alpha$ and $(a_k,1) \rightarrow (a_k,1) \circ (a_{k+1},1) = M(a_k) \circ M(a_{k+1})$

Periods of a 4-polytope formed from iteration steps k= 2^{2^i} are caused by periods {ā} of a cycle {a}, e .g. {11} has period 2. The inner structure of the 4-polytope is inaccessible (or SE(3) kinematics of four steps). Similarly, the inner structure of the surface δXs can be very complicated but four (three) ramification points exist.

At iteration steps k=2^{2^i} (envelopes) a fast algorithm exists analogous to fast multiplication algorithm for large integers. A Discrete Fourier Transform (DFT) is a highly composite Mersenne Number Transform (MNT(k)) of modulus $M_{2^k} = 2^{2^k} - 1$.

If $\exists C_M(3)$ with constants K_1, K_2, K_3 a Signal Processing (SP) includes

- Binary representation for a Power Integral Base (PIB) Z and MNT(K_1)
- a decomposition via the Chinese remainder theorem (CRT(K_2 -1)) with K_2 -1 coprime divisors K of $M_{2^{K_2}}$
- Fermat Number Transform (FNT (K)) with Fermat number K of modulus $F_k = 2^{2^k} + 1$
- Mersenne Number Transforms (MNT (K_3)) for highly composite 2- power cyclotomic fields [14].

One of the first algebraic spinor theories is based on hyperelliptic ϑ(u,L) and quaternions q which do not explain bi-spinor representations ψs [5]. Here ϑ(u,L) and quaternions q are embedded in subsequent μs^2 foliations with imaginary units i(FNT(l))= $2^{2^{l-1}}$ which explain four independent complex components ψ_s.

1. Epipolar geometry and coordinates

A point in space X ∈ P^3 imaged by n cameras C^i (i =1,..., n) with x ∈ P^2 via projections X= Px having 4 rows and 3 columns has matching constraints in computer vision [15]. The joint image Grassmann tensor $G^{abcd} = P_A^a P_B^b P_C^c P_D^d \varepsilon^{ABCD}$ forms fundamental, trifocal and quadrifocal tensors for n=2,3,4 cameras having 3^n parameters. Epipolar geometry yields

$$T = \varepsilon_{abe}\varepsilon_{cdf}x^e x^f G^{abcd} = 0,$$
$$T_{ij} = \varepsilon_{eab}\varepsilon_{fci}\varepsilon_{gdj}x^e x^f x^g G^{abcd} = 0$$
$$T_{ijkl} = \varepsilon_{eai}\varepsilon_{fbj}\varepsilon_{gck}\varepsilon_{hdl}x^e x^f x^g x^h G^{abcd} = 0 \quad (1.1)$$

which must hold for each of the 3^2 and 3^4 combinations ij and ijkl for n=3 and n=4, respectively. Repeated indices a, b and A, B etc. denoting homogeneous variables in P^2 and P^3, respectively, imply summation. There are only 29 algebraically independent tensor components in total. Calibration of X scenes $\in P^3$ requires four cameras. An over constrained system T, T_{ij}, T_{ijkl} minimizes the vectorization vec G^{abcd} of the Grassmannian within a linear least-squares algorithm [15,16]. A point AX depends on 32 parameters e. g. for a complex matrix A. Compared to 29 parameters of the Grassmannian G the question arises whether general X are determinable.

Linear fractional substitutions of x and X yield congruence relations in (1.1) with respect to

$$Q_{abs} = x^2 + y^2 + z^2 \quad (1.2)$$

and with respect to

$$Q_{abs}(r) = x^2 + y^2 + z^2 - r^2 \quad (1.3)$$

being calibrations in projective space. Thus, rational solutions for space points AX and AX' are provided with systematic errors ε_{focal}

$$\varepsilon_{focal} = \min\left(lT + l_{ij}T_{ij} + l_{ijkl}T_{ijkl}\right) \quad (1.4)$$

caused by optimizing calibration and currents X-X'.

A metric space calibrated in ℚ satisfies a triangle relation with a Cayley- Menger determinant of rank 3 [17]. A rationalized triangle leads to elliptic and hyperelliptic theta functions [18]. The differential domain with affine connection with torsion having no symmetries on its 4^3 indices must be separately discussed.

If detA≠0 the substitution X' =AX is well defined for X ∈ P^3 and for X∈$\mathbb{Q}^{3,1}$. Scene reconstruction is possible in computer vision, its complexity is high whereas the information current is low (unique solution).

If det A=0 critical configurations for projective reconstruction with ambiguous correlations X, X', ⋯ ,X_∞ appear [19]. Scene reconstruction is not possible, its complexity is low (Gauss fluctuations) whereas the information current is high (∞ solutions). Thus, quantum states and matter with charge and mass are caused by ambiguous correlations of non- unique X with singular A where det A=0.

The error term ε_{focal} in AX=ε_{focal} with rational AX is expressed by theta functions. Kummer surfaces K with det K(x) =0 and Weddle surfaces𝕎 with det WX=0 in §16 K(x) and W(X) have matrices linear in x=(x, t=1) and X, respectively [9].

Rational X∈ℚ of elliptic theta functions are iteratively determined via a self- consistent universal covering. $u[K\ \lambda[gu, L]$ A one- dimensional uniformization parameter $u_{si}^{\circ k}$ in Weierstrass σ- relations $\Sigma\sigma(u_1)\sigma(u_2)\sigma(u_3)\sigma(u_4)$ depends on iteration index k, parameter i = 1,2,3,4 , index s= 1,2,3,4 of a branched covering δX of a genus 1 Riemann surface with quarter period K of the lattice L ,Legendre parameter λ , modular units g[u,L].

Addition step k, k+1 and k+2 with u, v, u∓v can be visualized by a Poncelet polygon in space leading to u, v ∊ qML with q∈ $\mathbb{Q}^2$. The idea is to iterate rational maps linear both in $u_{si}^{\circ k+1} \leftarrow q u_{si}^{\circ k}$ where rational q are pseudo- random and would result from undecidable Diophantine equations.

Already a Lattès map as a doubling map 2u← u as an exactly solvable tent map T_2 yields fourth order rational quotient functions and a sixth order polynomial . A tent map T_c can be chaotic [7,20-22].

An elliptic curve E_λ over a subfield K of C has complex multiplication (CM) if the ring of endomorphism of E_λ end (Eλ/K) ≅ {M∈C:ML⊂L} ≠Z is nontrivial. The multiplier M is understood as a complex constant or a fractional substitution which is not an integer multiple of a matrix in SL(2, Z).

CM of E_λ is singular if M ∊ ℚ[√‾d] for an imaginary quadratic field with class number h_d =1 with e_d = (3; 2;1) and discriminant d= {-3;-4; -(7,1,19,43,67,163,49)}.

The imaginary quadratic field M ∈ ℚ[√‾d] describes the normal field ℕ [√‾d] = K K' K'' of a monogenic cubic field K(∂) with irrationality ∂ and its conjugates K',K''. For singular CM a lattice L ∊ K (∂) is homomorphic to an imaginary quadratic field M ∈ ℚ[√‾d].

Claim 1

One- dimensional interval $I_{k,k+1} = I(z^{\circ k}, z^{\circ k+1})$ and tangent spaces

Tangent spaces or asymptotic lines X ∊ P^3, x ∊ P^2 are mapped to one-dimensional intervals $I_{k,k+1}$ via $x_i^{(d)} \rightarrow \frac{x_i^{(d-1)}}{x_0^{(d-1)}}$ relating $P^d \rightarrow P^{d-1}$ with homogeneous variables. A map $I_{k,k+1} \rightarrow I_{k+1,k+2}$ of the interval to itself is chaotic if variables z_i are on inflection tangents (flex lines) X ∊ P^3, x ∊ P^2.

Proof

Let the interval [0,1] = $I_0 \cup I_1$. Homogeneous variables X ∊ P_3, x ∊ P_2,z ∊ P_1 of corresponding polynomial equations F(X)=0, F(x)=0 F(z)=0 can be related to each other if Hessian matrices H(F)=0 vanish [23]. A vanishing Hessian H(F)=0 is related to asymptotic lines as lines of zero curvature and singular points and reduces d dimensions to d-1 dimensions. In case of d=1 for a cubic polynomial F(z) one gets an equianharmonic E_λ. Flex lines for binary variables z (asymptotic lines) are defined if three points i, j=1,2, 3 satisfy [24].

$$\begin{vmatrix} \delta_{21}\zeta' & \delta_{31}\zeta' \\ \delta_{21}\zeta'' & \delta_{31}\zeta'' \end{vmatrix} = 0 \qquad (1.5)$$

where $\delta_{ij}\zeta = \zeta(u_i) - \zeta(u_j)$. Equation (1.5) holds also if the Weierstrass ℘-function ℘(u, L) is replaced by a fractional substitution $\gamma_H \wp(u, L)$. Condition (1.5) is equivalent that three points z_k , z_{k+1} , z_{k+2} are on different sites of E_λ flex lines or equivalently, if a simplest cycle exists

$$z_k = z_{k+3} or\ z_{k+1} < z_k < z_{k+2} or\ z_{k+2} < z_k < z_{k+1} \quad \text{for}$$

an interval $I_{k,k+2}$. The existence of this simplest cycle yields a chaotic map [25]. Extending the matrix (1.5) denotes the addition theorem (4.1) below ∀k. As a consequence (1.5) is equivalent chaos for $k \rightarrow \infty$ where $z_{k+1} - z_k$ is proportional to $z_{k+2} - z_{k+1}$ giving Feigenbaum constants.

Claim 2 on cycle constants K_1, K_2, K_3

Where a cubic irrationality ∂ where $z^{\circ k} \in K(\partial)$ requires two generators μ and μ' of 2- power cyclotomic fields to describe z via BCF with M (a_1,a_2) [26]. Then z as a two- dimensional DFT of itinerary Σ_2 (s mod 2: $z^{\circ k} \in I_s$) of $z^{\circ k}$.

$$\sum_{lk} = 2^{-2^{K_1}-2^{K_2}} \sum_{k'=0,l'=0}^{M_{2^{K_1}} M_{2^{K_2}}} 1^{ll' 2^{-2^{K_1}} + kk' 2^{-2^{K_2}}} \sum_{l'k'}$$

has cycles in the shift map : $\sigma \circ \Sigma_2 \{s_i\} = \Sigma_2 \{s_{i-1}\}$.

Congruences of cycles $C_M(n)$ yield {modulo $M_{2^{K_1}} \circ \ldots \circ$ modulo$M_{2^{K_2}}$} for n= 1, 2 , 3. A subsequent map $C_M(n) \circ C_M(n) \circ \ldots$ forms a multidimensional MNT with constants $K_1,\ldots,K_n$.. For K_n ≦8 the map $z^{\circ k}$ is ambiguous.

Proof:

Ambiguous correlations X and X': AX=0, AX'=0 obey a quadratic map of the interval $I_{k,k+1} \rightarrow I_{k+1,k+2}$ where z=℘(u) transforms according to black- box map (2.8) or (2.9). Periods (2.8) of Hermite transformation (2.4) $\gamma_\wp z^{\circ k} = \mu z^{\circ k} = z^{\circ k+1}$ and $\gamma_{\wp'} z^{\circ k+1} = \mu' z^{\circ k+1} = z^{\circ k+2}$ are roots of the characteristic equation $\det(\gamma \wp M(a) - z) = 0$.

The z- degree of $z^{\circ k}$ is 2^{2^k} three congruences $C_M(3)$ in a CCF matrix M(a_1, a_2, a_3) in (3.2) can be approximated as a direct product by abelian extensions of the rational number field. However,

$$M(a_1, a_2, a_3) \neq \gamma_\wp \otimes \gamma_{\wp'}$$

BCF and CCF in $z^{\circ k+2} \leftarrow \Gamma \circ z^{\circ k}$ require a crossing term in the black box operator Γ with at least two CF periods. As a result Hermite's problem for ∂ is ambiguous. However $z^{\circ k}$ is highly composite and allows a CRT(K_1- 1) decomposition of $\binom{k_1-1}{2}$ pairwise coprime divisors and a fast FNT(K_1) followed by a fast MNT(K_2) in the limit $K_2 \rightarrow \infty$

Whereas a multiplication of polynomials f requires deg2f steps a fast 2- radix DFT requires deg f log deg f steps.

Cycles imply the existence of elliptic units $g(\mathbf{q}_s\boldsymbol{\omega}, L)$ (3.6) with $u_{si}^{\circ k} = \mathbf{q}_s\boldsymbol{\omega}, r, s \in Q^2$ as units of the modular group $\Gamma(N) \in N$.

Maps $\gamma_\wp$ are as well quadratic and linear substitutions of cubic roots leaving (3.8) invariant. The Legendre module λ of E_λ depends on uniformisers of the modular group (N) enveloping (3.6) [10].

Weierstrass relations $\Sigma\,\sigma(u_1)\,\sigma(u_2)\,\sigma(u_3)\,\sigma(u_4) = 0$ are invariant if four parameters suffer substitutions $u_{si} \to u_{si} + \frac{1}{2}\hat{U}L$ with a matrix Û of two columns and four rows having 28 values 0 or 1[27].

Two iterates $u_{si}^{\circ k}$ and $u_{si}^{\circ k+1}$ differ by two ½ ÛL values.k iterations differ by 2^k values ½ ÛL which yields congruences, i.e. maps are ambiguous for $C_M(i)$ if $K_i \leqq 8$. Substitution matrix $\hat{U}_{k\to k+1}$ u have rank four. The itinerary $\Sigma_2$of $z^{\circ k}$ is defined by Σ_2 {s mod 2: $z^{\circ k}\in I_s$ } [22] .

The shift map σ is defined by $\sigma \circ \Sigma_2\{s_i\} = \Sigma_2\{s_{i-1}\}$. A DFT of a is defined by

$$\tilde{a}_l = 2^{-2^L}\sum_{l'=0}^{M_{2^L}} 1^{ll'2^{-2^L}}\,\Sigma a_{l'}$$

as a congruence module M_{2^L} Two congruence moduls $M_{2^k z}, M_{2^k u}$ yield 2D DFT,

$$a_{lk} = 2^{-2^{K_x}-2^{L_u}}\sum_{k'=0,l'=0}^{M_{2^{K_x}}M_{2^{K_u}}} 1^{ll'2^{-2^{K_u}}+kk'2^{-2^{K_x}}}\,a_{l'k'}$$

Power-2 cyclotomic fields allow a fast decomposition of $u_{si}^{\circ k}$ in terms of congruence modules with Fermat number F_t via K_1 =8 in

$$M_{2^k} = \prod_{i=1,\dots,k-1} F_i \qquad (1.6)\ [14]$$

A peculiarity is that 2-power towers of bases 3 and 2 are generators mod F_t. Number 2 and 3 are primitive roots of unity of FNT (t) for t≦4 which do not split within $Q[\sqrt{d}]$. Roots of unity correspond to ray class fields of lattices L establishing a connection between μ and μ' and modular units $g(\mathbf{q_s}\boldsymbol{\omega}, L)$ in (3.6). 2- power cycles create the simplest cycle $1^{1/3}$ [25].

A Lattès map $i(u) \circ u = \alpha u + \beta, \alpha, \beta \in C$ is understood in terms of CM fields, i. e. $i(u) \circ u \in ML$. In distinction i(u) is identified with a Poncelet involution $i^2(u)$ = 1.

A chaotic map exhibits 2^k periodic cycles: $\exists C_M(3)$. Pseudo- random number $z = \wp(u_{si}^{\circ k})$ of one-dimensional map γ_H are exactly solvable maps of E_λ within interval [0,1] [13]. CM transfers the endomorphism for a complex constant $M \in C$ to $z = \wp(u_{si}^{\circ k}, ML)$ implying existence of PIB [11, 28]. A PIB regularly maps I⟶ I' of exactly solvable chaos.

A sum π/ω $\Sigma \mp u_1 \mp u_2 \mp u_3 \mp u_4$ corresponds to spherical triangles of S^2 [29]. The coefficient π/ω is irrational and depends on a ternary black-box map for $u_{si}^{\circ k}$, $u_{si}^{\circ k+1}$ and $u_{si}^{\circ k+2}$ as follows

$$(a_{(2k+2)}, b_{(2k+2)}, a_{(2k+3)}, b_{(2k+3)}) = \Gamma_{k+1\leftarrow k}\left(a_{2k}, b_{2k}, a_{2k+1}, b_{2k+1}\right)^t \quad (1.7)$$

with a black- box matrix $\Gamma_{k+1\leftarrow k}$ of four columns and four rows.

An arithmetic-geometric mean algorithm of Gauss (agM) $a_{2k+1} = \sqrt{a_{2k}}, b_{2k+1} = \sqrt{b_{2k}}$ BCF (Jacobi algorithm) with a_{2k+1}=int(a_{2k}) , b_{2k+1}=int(b_{2k}) depends on three or four columns. The agM limit $k \to \infty a_\infty = b_\infty = \omega \in K(\partial)$ yields the Dedekind eta function η(ω) and Weber- Schlaefli invariants f (ω), f_1(ω) as

$$a_k = \vartheta_{00}^2(2^k\omega) = \eta^2(2^k\omega)\, f^2(2^k\omega)$$

and $b_k = \vartheta_{01}^2(2^k\omega) = \eta^2(2^k\omega)\, f_1^2(2^k\omega)$ [26,30]

In dependence on initial a_0, b_0 values a limit is reached $a_\infty = b_\infty \in \omega_{h,} K(\partial)$ or π / ω_h where ω_e or ω_h are equianharmonic or harmonic (lemniscate) constants. A ternary BCF limit for $\partial = 2^{1/3}$ yields period 2 sequences a = { 1 (1 2) } and b = { (1 0) }. A ratio π / ω_h calibrates $\Sigma \mp u_1 \mp u_2 \mp u_3 \mp u_4 \in K(\partial)$ between T and S^2 where $\Sigma \mp u_1 \mp u_2 \mp u_3 \mp u_4 \simeq \pi \in S_2$. The calculation π/ω_h requires a 4- component algorithm (1.7) in case of harmonic E_λ and ω_h. An infinite expansion in (1.7) goes over 2-power maps of f(ω) which can be approximated by cyclotomic units μ.

Below this μ- expansion is confirmed by a hyperelliptic doubling map which relates spinor states to iterations of Weber- Schlaefli invariants f(ω).

According to (4.1) this expansion holds also for u,ζ(u) and℘(u) ∈ K(∂) [31]. For a fractional substitution $\wp(u) = \gamma^\circ \vartheta^2(u)$, one gets also $\vartheta^2(u, L) \in K(\partial)$ [31].

Claim 3: Cyclotomic units and Riemann surfaces δX

Cyclotomic unit's μ and μ' couple hyperelliptic surfaces on layers

$$\delta X = \{X_2(\mu) U X_2(\mu')\} = \{X_1(M_0) U X_1(M_1) U X_1(M_{0'}) U X_1(M_{1'})\}$$

as elliptic curves E_λ and $E_{\lambda'}$ with variable Legendre module λ and λ', CM multiplier Ms, and uniformization parameter u_s where s=0, 1, 0', 1'.

Proof: Rational self- maps of the T = X_1 to itself are constrained by the Hurwitz automorphism theorem [32].

$$\sum_{i=1}^{w} 1 - r_i^{-1} = \chi(\mathbb{X}_0) = 2$$

with Euler- Poincaré characteristic χ. The number of branch points w is identical to the number of generators as r_i th roots of unity in δX.

The branched covering δX are four layers as tori with four branch points {r_i}={2,2,2,2} or three branch points {r_i}={2,3,6} , {2,4,4} or {3,3,3} [33,7].

Elliptic (modular) units (3.6) of E_λ, $E_a . \lambda = \frac{e_1 - e_2}{e_1 - e_3}$ are indexed by generators g (μ,L) . Homogeneous functions are invariant with respect to the multiplication of a lattice period ω by M, e. g. the Weber τ- function $\tau = G^{2e_d} (-1)^{e_d} \wp^{e_d}$ with $G = \frac{2^7 3^4}{\Delta}(3g_2 g_3, 2g_2^2, 6^2 g_3)$. Here g_2 and g_3 are Weierstrass invariants. The addition theorem (3.8) formulated in terms of the invariant ℘- function (3.7) leading to (3.8) holds on δX.

The complexity for uniformizing rational x is comparable to Diophantine solutions [18]. Rational x require lattices L as a genus g dimensional polytope reducible to a torus T in one complex dimension with L period ω∈ K (∂). Then an algebraic approach to epipolar geometry in P^2 is uniformized by elliptic Weierstrass functions being a Lagrange parameter in matrices of rank 3 [34]. However a pencil of two P^3 quadrics yields four P^2 conics depending on 20 parameters [35]. Already a pencil of three quadrics yields hyperelliptic solutions [9].

A determinant A vanishes if three (if A ≠ A^t four) first minors vanish. Fundamental matrix, trifocal and quadrifocal tensor depend on first minors of A. Rational singular points X or x of ambiguous configurations K(x)X=W(X)x=0 correspond to the invariant- theoretic expression

$$2(\theta\phi)^2 (b\theta)(b\phi) - a_\theta^3 a_\phi^3 = 0 \tag{1.8}$$

where

$$\boldsymbol{x} \simeq (b_0^2, -b_0 b_1, b_1^2), X \simeq (\theta_0^3, -\theta_0^2\theta_1, \theta_0\theta_1^2, -\theta_1^3) \simeq (\phi_0^3, -\phi_0^2\phi_1, \phi_0\phi_1^2, -\phi_1^3)$$
(1.9)

θ, ϕ, and b are roots of a sextic polynomial $f_n(\theta) = \sum_{i=0,\dots,n} a_i \theta^{n-i}$ (n=6) splitting into $f_6(\theta,\phi) = a_\theta^3 a_\phi^3$. (1.10)

Here $a_i = a_0^{n-1} a_1^i$, $a_\theta = a_0\theta_o + a_1\theta_1$, $(a\theta) = a_0\theta_1 - a_1\theta_o$, K(x)X=W(X)x, detK(x)=0 and detW(X)=0

The notion γ- invariant is used regarding groups GL (2, Z), GL(2, Q), GL(2, C) generalizing modular invariance.

Equation (1.8) is equivalent to

$$\frac{1}{3} D_c^4 = \left[(ab)^2 a_c b_c + a_c^2 (aD)\right]\left[(ab)^2 a_c b_c - a_c^2 (aD)\right] \tag{1.11}$$

acting onto sigma functions σ(u)σ(u') where (aD) = $a_0 D_1 - a_1 D_0$ and

$$D_0 = \frac{\partial}{\partial u_0} - \frac{\partial}{\partial u_{0'}}, D_1 = \frac{\partial}{\partial u_1} - \frac{\partial}{\partial u_{1'}} \tag{1.12}$$

involves replacing u' by u in σ(u)σ(u'). In distinction to §13 μ the invariant differential operator (1.11) [9]

$D_c^4 = (D_0 c_c + D_1 c_1)^4$ is factorized into two linear parts (aD). Rational solutions X(θ) ∈ P^3 ∈Q^4 of symbolic invariants (1.8) and (1.11) are hyperelliptic theta functions

$$\vartheta\begin{bmatrix} g \\ h \end{bmatrix}(u,\omega) = \sum_{n\in\mathbb{Z}^2} 1^{\left(u(n+g) + \frac{\omega}{2}(n+g) + h(n+g)\right)} \tag{1.13}$$

for rational characteristics g_0, g_1, h_0, h_1 ∈ Q^4 . It is of interest that five arbitrary parameters (coordinates) a_1, a_2, b, c_1, c_2 enter solutions of (1.8) and (1.11) leading to sigma functions

$$\sigma\begin{bmatrix} g \\ h \end{bmatrix}(u,\omega) = e^{au+b} \vartheta\begin{bmatrix} g \\ h \end{bmatrix}(u+c,\omega) \tag{1.14}$$

A relation between even hyperelliptic σ(u), ϑ(u) and zeta function Z(u)

$$\sigma\begin{bmatrix} g \\ h \end{bmatrix}(u,\omega) = \exp\left(\int du_s Z_s\right) = \exp\left(\frac{-1}{20\vartheta}\sum_{ss'} u_s u_{s'} \vartheta_{ss'}^{"}\right) \vartheta\begin{bmatrix} g \\ h \end{bmatrix}(u,\omega) / \vartheta \tag{1.15}$$

implies a quadric u- functional in the exponent which transmits to Jacobi theta and elliptic sigma functions

$$\sigma(u,L) \simeq \exp\left(\frac{\zeta(\omega_2/2)u^2}{2\omega_2}\right)\vartheta_i(2Ku,\lambda) \text{ [36]} \tag{1.16}$$

for quarter periods K and K' , $\omega_2 = \frac{1}{2K}$ and Jacobi theta ϑ_0, ϑ_1, ϑ_2, ϑ_3. Modular units of (1.16) exhibit a transient response in dependence on λ around the minimum λ_0 of the quadratic exponent. Here the multiplier M in CM depends on Legendre parameter λ of E_λ via M(λ) or λ(M), respectively. For arbitrary characteristics g, h elliptic σ(u) depend on Jacobi zeta functions $\zeta_1,\dots,\zeta_6$ [37]. The appropriate description of σ(u) is given by the Heuman lambda Λ(u,λ) function.

$$\Lambda(u,\lambda) = \frac{u}{K'} + \frac{2K}{\pi}\frac{\partial \ln \vartheta_4(u,\lambda')}{\partial u} \tag{1.17}$$

with $\vartheta_4 = \vartheta_{01}$ and λ'= 1-λ. For elliptic lattices $L_0 \oplus L_1$ (1.15) is proportional to

$$\prod_{s=0,1} e^{\sum_{l=0}^{k} S\left(u_s^{\circ l}, L_s\right)} \tag{1.18}$$

and defines an elliptic S- matrix for the k[th] map $u^{\circ k}$, introduced here as follows

$$S\left(u^{\circ k}, L_s\right) = \frac{\delta}{\delta k}\int_0^{u^{\circ k}} dv \Lambda(v,\lambda) \tag{1.19}$$

Equations (1.11) of high generality put a problem: To obtain a theory of differential equations for n+1 Weierstrass functions for i =0 ,1,...,n

$$\wp_i^{(n)}(u) = \wp_{i_1 \dots i_n}(u) = \frac{-1}{\sigma^2}\left(\prod_{k=1,\dots,n} D_{i_k}\right)\sigma(u)\sigma(u') \tag{1.20}$$

sigma functions σ(u) have properties which a priori we know it to possess [38]. For genus g=3 with i_n=(0,1,2) the system is equivalent to Korteweg-de-Vries (KdV) and Kadomtsev- Petviashvili (KP) equations [39].

W and K are quasi-two- dimensional rational surfaces, z in $\boldsymbol{x} = (x_i) = (\wp_i^{(2)})$ are rationally expressed by x and y via $\wp_2^{(2)} = (\wp_0^{(3)})^2 - f_3(\wp_0^{(2)}, \wp_1^{(2)})$ with a cubic polynomial f_3. The aim is to find rational x, X on surfaces K and W as complex multiplication (CM) on E_λ with pencil parameter $\wp_i^{(2)}$. Then rational solutions are bitangents or lines on cubic surfaces or trivial lines on quadrics and ruled surfaces. Binary x, y and z via $x=x_1/x_2$, $y=y_1/y_2$, $z=z_1/z_2$ generate ternary E_a for x_1, x_2 and x_3 and y_1, y_2 and y_3 and z_1, z_2 and z_3. Rational solutions of (1.8) yield ruled surfaces as pencils of lines in mutually dependent projective spaces P^1, P^2 and P^3 as a result of homogenization variables. In theta functions of KdV and KP equations u- parameters are viewed as coordinates. One addition step k on E_λ denotes a line between $u^{\circ k}$, $u^{\circ k+1}$ and $u^{\circ k+2}$. For k→∞ the index k ∈ N gets complex k∈ C. Coordinates are introducible if $u^{\circ k}$ are infinitely differentiable $u^{\circ k}$∈ C∞ with respect to k as shown below.

The main proposition of the presented paper is the existence of infinitely differentiable $u^{\circ k}$ in the limit $k = 2^{2^i} \to \infty$ where k gets complex due to modular congruences.

p-functions differing from $\wp$ by a $\gamma_\wp$ substitution p= $\gamma_\wp \wp$ are equivalent. Modular units g(u,L) are modular invariant sigma functions multiplied by $e^{q_s \omega u}$ .However, the exponent is quadratic with respect to q_s = (r, s)∈Q^2 which indicates that the Heuman Lambda function (1.17) should describe the asymptotic behaviour. Rational solutions for elliptic units on E_λ imply that the first derivative $\frac{\partial}{\partial u_s}$ is expressible by $A_{q_s} = \zeta\left(q_s \omega\right) - \omega\eta$ as a linear shift of the Weierstrass zeta function by **ωη** where $\boldsymbol{\eta} = \zeta(\boldsymbol{\omega} / 2, ML), \boldsymbol{\omega} = \left(\omega_1, \omega_2\right), \mathbf{q}_s \boldsymbol{\omega} = r\omega_1 + s\omega_2$ [40].

Already a second derivative brings insurmountable problems to determine the proportionality coefficients between $\wp = A_\zeta \zeta$ and $\wp' = A_\wp \wp$. Modular units satisfy

$$\eta^{2N}(\omega) = \prod_{q \in \Gamma(N)} g\left(q\omega, \mathrm{L}\right)$$

for elements of modular group Γ(N) and

$$g\left(q\omega, \mathbb{L}\right) = \frac{\prod \eta(q'\omega)}{\prod \eta(q''\omega)} \qquad [41]$$

for modular congruences q', q'' of q where η(ω) is the Dedekind eta function [42].

Powers $\eta^d(\omega)$ of η(ω) for dimensions d of relevant Lie algebras yield determinant evaluations in terms of integers and $q = 1^{\frac{\omega_1}{\omega_2}}$ series [43-45].

To find rational q_s with N ∈ N where ω ∈ QML, M ∈ Q [$\sqrt{}$d], L∈K [∂] implies to solve a Diophantine equation. However, Hilbert's tenth problem is not decidable, i. e. a general algorithm does not exist or its complexity is too high.

Therefore, pseudo- random values $A_{q_s} = \zeta\left(q_s\omega\right) - \omega\eta$, i.e. values of the operator $\frac{\partial}{\partial u_s}$ are linked to a cycle $C_M(n)$ whereas higher derivatives of u_s are determined via iteration.

A linear expression of ζ, ℘, ℘' in terms of derivatives $\frac{\partial}{\partial u_s}$ exists for hyperelliptic sigma functions σ(u)σ(u) with subsequent limit u'→u which serves as the base of the present approach.

If ∃ q_0, q_1, q_0, q_1 then (1.12) takes the form

$$D_0 = \left(q_0 - q_{0'}\right)\omega, D_1 = \left(q_1 - q_{1'}\right)\omega \tag{1.21}$$

CM implies a direct sum of lattices $L_0 \oplus L_1$ and $E_{\lambda_0} \oplus E_{\lambda_1}$ It should be noted that symbolic (1.8) and (1.11) contain coefficients $a_i = \alpha_0^{n-i}\alpha_1^{i}$ whereas rational x and X in(1.9) are Jacobi theta functions Then Riemann surfaces are layers of tori as more general Riemann surfaces X.

2. Twisted cubic and coordinates

Coordinates imply a $C_M(1)$ as an invariance with respect to $M(a_1)$ substitutions. The elliptic addition theorem creates an invariant equation. If hyperelliptic thetas are reducible to elliptic theta a fractional substitution changes planes $\boldsymbol{x} = \left(x_i\right) = \left(\wp_i^{(2)}(u)\right) \in P^2$ into space points $X = \left(X_i\right) = \left(\wp_i^{(3)}(u)\right) \in P^3$ being proportional to second and third derivatives $\wp^{(2)}$ and $\wp^{(3)}$ of lnσ(u) for x=(x, 1) .

Similarly for elliptic theta x_i ∃ γ, u_i , i=1, 2 with $\wp'(u_i) = \gamma\wp(u_i)$ then

$$x_3 = \frac{\det\gamma}{\left(cx_1 + d\right)\left(cx_2 + d\right)} - x_1 - x_2 - a_1$$

It is assumed that also $u_0 = \gamma u_1$∈Q ML

In terms of a hyperelliptic differential

$$d = du_0 \frac{\partial}{\partial u_0} + du_1 \frac{\partial}{\partial u_1} \tag{2.1}$$

osculating planes det(X, X', dX ,d^2X) =0 as tangent planes yield a condition of zero curvature if

$$\begin{vmatrix} x & dx & d^2x \\ y & dy & d^2y \\ z & dz & d^2z \end{vmatrix} = 0 \tag{2.2}$$

where columns in rank- 4 matrix in §40 μ are multiplied by du_s. Hyperelliptic reduction yields $P^3 \to P^2$ and a similarity between asymptotic lines (2.2) and the addition theorem (4.1) below.

The tangent plane of det K(x)= 0 is given by (3.1) or

x(u)y(v)-x(v)y(u)+z(v) – z(u)=0. With (1.9) one gets a quadratic equation

$$\theta^2 - \theta\wp_{11}\left(2u\right) - \wp_{01}\left(2u\right) = 0 \tag{2.3}$$

where $\theta_1 + \theta_2 = \wp_{11}\left(2u\right)$ and $\theta_1\theta_2 = \wp_{01}\left(2u\right)$ with $\theta = \frac{du_0}{du_1}$ and roots $\theta_{1,2}$. Equation (2.3) is equivalent to a rationalized condition (1.9) for chords of a conic in P^2.

A subsequent quadratic map $x \to \theta^2$ yields rational hyperelliptic $\wp\left(2u\right)$ reducible to elliptic theta functions $\partial(u, L) \simeq \theta = \frac{\delta u_o}{\delta u_1}$.

As solutions for a quartic polynomial. Hermite substitutions $\gamma_\wp$ of roots leave a cubic polynomial $f\left(z\right) = a_z^3 = 0$ invariant where

$$\gamma_\wp\left(t\right)z = \frac{f\left(t\right)}{t - z} - ⅓ f'\left(t\right) . \tag{2.4}$$

One has det $\gamma_\wp$=f(t) and deg $\gamma_\wp$= 3. Thus, if the cubic residue symbol $\left[\frac{t}{p}\right]_3 = 1$ one gets rational $\gamma_\wp$ ∈Q for congruence groups modulo p.

Star generation implies four parameters θ, θ', θ'', θ''' of rational points X(θ) forming a twisted cubic curve C_{tw} in space a chord (X(θ'') , X(θ''') and a star point X(θ) or X(θ') span two planes π: X(θ), X(θ''), X(θ''') and π': X(θ'), X(θ''), X(θ''').

$\theta'' + \theta''' = \wp_{11}\left(2u\right)$ and $\theta''\theta''' = \wp_{01}\left(2u\right)$ .Then roots θ'', θ''' of equation (2.3) represent two planes

$$\pi : \left(\boldsymbol{x}\left(\theta'' = \theta_0''/\theta_1''\right)\right) = \left(\theta_0''^2, -\theta_0''\theta_1'', \theta_1''^2\right)$$

and $\pi' : \left(\boldsymbol{x}\left(\theta''' = \theta_0'''/\theta_1'''\right)\right) = \left(\theta_0'''^2, -\theta_0'''\theta_1''', \theta_1'''^2\right)$ forming C_{tw} , respectively.

Absolute quadrics Q_{abs} (1.2) or (1.3) result from rational (1.9) and are invariant with respect to $\gamma_\wp$ ∈ SL(2, Z)

$$\begin{pmatrix}\theta_0 \\ \theta_1\end{pmatrix} \leftarrow \gamma_\wp \circ \begin{pmatrix}\theta_0 \\ \theta_1\end{pmatrix} \tag{2.5}$$

or

$$\begin{pmatrix}\theta_0\\ \theta_1\end{pmatrix} \leftarrow \gamma_{\wp} \circ \begin{pmatrix}\theta_1\\ \theta_0\end{pmatrix} \tag{2.6}$$

$\gamma_{\wp} \in SL(2, Z)$ are Cayley- Klein parameter of a classical spinning top here Hermite transformations $\gamma_{\wp}$ leave invariant a cubic polynomial [46].

At step k a cubic invariant appears subsequently rationalizing $\theta_0^{\circ k} \rightarrow \left(\theta_0^{\circ k+1}\right)^2 - \left(\theta_1^{\circ k+1}\right)^2$, $\theta_1^{\circ k} \rightarrow 2\theta_0^{\circ k+1}\theta_1^{\circ k+1}$ for k, k+1

Starting from

$$x(q) = \left(\wp_i^{(2)}(q),1\right) \tag{2.7}$$

also $\theta^{\circ k+1} \simeq \vartheta(u, L)$ and

$$\wp_i^{(2)}\left(\gamma_{\wp}\circ\gamma_{\wp}\circ u\right) \leftarrow \gamma_{\wp} \circ \gamma_{\wp} \circ \wp_i^{(2)}(u) \tag{2.8}$$

with a black- box map Γ

$$x^{\circ k+2} = \Gamma_{box} x^{\circ k} \tag{2.9}$$

with a fixed equation.

An identical vanishing of hyperelliptic sigma functions of l.h.s of (3.1) is provided which is equivalent to (1.5) and (4.1).

A black- box map (2.9) has lower complexity than the doubling map of hyperelliptic $\wp$ in §40 of as well Lattès maps for elliptic theta if μ steps $k = 2^{2^i}$ are regarded [33].

Fixed points of $\gamma_{\wp}$ and $\gamma_{\wp}$ in (2.8) correspond to a cubic invariant polynomial a_0^3 or $f(\theta^{\circ k+1})$

$$\theta^{\circ k+1} \leftarrow \gamma_{\wp} \circ \theta^{\circ k} \tag{2.10}$$

(2.5) and (2.6) describe exact classical spinning top precession or incompressible fluid dynamics for continuous time t ∊ L. Geodesic motion on tessellations of the hyperbolic plane H is quasi- ergodic. Cubic reciprocity in (2.4) with $\gamma_{\wp}$ ∊ Q creates a pseudo- random component and chaotic tessellations of H.

Congruences of $\gamma_{\wp}$ are roots of unity $1^{1/N}$ of modular congruence groups Γ(N) [41,46, 47].

Then E_λ modular units undergo quadratic maps (see claim 2). Hyperelliptic thetas are elliptic theta with variable modulus λ which has been drawn [9]. Iterates (2.8) are regular maps or pseudo-random maps if γ- iterates constitute a principal ideal domain (PID) [11]. Subsequent quadratic substitutions yield at k⟶∞ the Diophantine equation.

$$\theta_1^4 - \theta_2^4 = \square \tag{2.11}$$

which has solutions for nine lattices L with class number h_d=1 [48]. where $\theta_{1,2}$⟶ Weber- Schlaefli invariant f (√d) [18]. For h_d = 1 (2.11) reduces to a cubic polynomial

$$\mathfrak{f}^3\left(\sqrt{d}\right) + 2E\mathfrak{f}^2\left(\sqrt{d}\right) + 2F\mathfrak{f}\left(\sqrt{d}\right) + 2 = 0 \tag{2.12}$$

with (EF) = (00),(11),(0-1),(10),(1-1),(32) $\in N^2$ [49].

3. Partition function and topological entropy

The surface K(x)X=W(X) x=0 can be formulated in terms of four points θ, θ', θ'', θ''' or u, v, u + v and u - v . It is equivalent to the for addition theorem in §37 of hyperelliptic sigma functions $\sigma(u)=\sigma(u_0,u_1)$

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma^2(u)\sigma^2(u)} = \sum_{i,j=1,\dots,4} x_i(u)F_{ij}x_j(v) = 0 \tag{3.1}$$

with $F=i\sigma_x \otimes \sigma_x$, Pauli- matrices σ_x , det F=1 , x=(x, 1) and $x_i(u_k) = \wp_i^{(2)}(u_k)$ I search Solutions of (3.1) as unimodular collineations of the Plücker matrix F by means of the CCF matrix

$$M(a) = \begin{pmatrix} 0 & 0 & 0 & 1\\ 1 & 0 & 0 & a_1\\ 0 & 1 & 0 & a_2\\ 0 & 0 & 1 & a_3\end{pmatrix} \tag{3.2}$$

depending on $\hat{t}_n$ a cyclic permutation matrix of order n which forms the PIB$1, \hat{t}, \dots, \hat{t}_n^{n-1}$. Rows of iterated matrices F=∏M(a_k, b_k, c_k) consist of

$$A_k = A_{k-1} + a_{1k} A_{k-2} + a_{2k} A_{k-3} + a_{3k} A_{k-4} \tag{3.3}$$

for A= $\mu, \zeta, \wp$, respectively. A step a_{1k+1}, a_{2k+1}, a_{3k+1} ⟵a_{1k}, a_{2k}, a_{3k} depends on $\left[\hat{t}_4, \hat{t}_3, \hat{t}_2\right]$ whereby (3.1) is viewed as a nilpotent operator. A $\hat{t}_3$ cycle at iteration indices k, k-1 and k-2

$$A_k = a_{1k} A_{k-1} + a_{2k} A_{k-2} + A_{k-3}$$

forms eigenstates of a matrix of third order

$$\left[x_k, x_{k-1}, x_{k-2}\right] = \prod_{i=0,\dots,k} M\left(a_{1i}, a_{2i}\right) \tag{3.4}$$

as a BCF matrix exhibiting partial cycles [50].

$$M(a) = \begin{pmatrix} 0 & 0 & 1\\ 1 & 0 & a_1\\ 0 & 1 & a_2\end{pmatrix}. \tag{3.5}$$

Rational solutions are defined by an identical vanishing of the R.H.S. (3.1) which can be regarded as a square of nilpotent matrix N_A of rank four. Unimodular collineations of N_A via M(a) and CCF (3.2) embedded into BCF (3.5) and CF constitute a Hermite problem of expressing cubic irrationalities [26,51]. Iteration steps k= $k = 2^{2^{2'}}$ lcorrespond to (3.3). To detect the simplest cycle 0=3 four points 0, 1, 2, 3 are needed. Thus, a CCF allows to detect all cycles as modular units in a cyclotomic unit with generator $1^{1/N}$[10] Matrix elements of γ are L- normalized Weierstrass σ- functions [11]. Generators for modular unit.

$$g\left(q_S\omega, L\right) = \Delta^{1/12} e^{-q_S\eta\cdot q_S\omega}\sigma\left(q_S\omega, L\right)$$
$$= \Delta^{1/12} e^{-q_S\omega q_S\eta - \int_0^{q_S\omega} dv\left(\zeta(v,ML) - \frac{1}{v}\right)} \tag{3.6}$$

with discriminant $\Delta = (2\pi)^{12}\eta^{24}\omega_2^{(-12)}$

Complex multiplication ML, M∊C transforms an elliptic invariant equation for j(L) into its Tschirnhaus resolvent. For h_d =1 the invariant j(L) ∊ N satisfies a linear equation. At h_d =1 the relevant quantity is the Weber invariant γ_2 or the Weber- Schlaefli invariant f(ω). Again the problem is reduced to a Tschirnhaus resolvent of a cubic equation (2.12). The Weber- Schlaefli invariant f(ω) suffers Hermite substitutions γ_f which are quadratic in f(ω). Normalized $\wp$- functions of the normal field N[√d] = KK'K'' are

$$P(u,L) = \left(\in\frac{\wp(u,L)}{\sqrt[6]{d(L)}}\right)^{e_d} \tag{3.7}$$

With a unit ∊ and e_d defined above.

Cyclotomic monogenic fields with units constitute envelopes of modular units (3.6) [52]. Normalized Weierstrass P - functions depend sensitively on regions of discontinuity of L. P- differences yield products of g(u)= g(u,L) for e.g. square lattices (e_d=2) and hexagonal lattices (e_d=3) in addition theorem (4.1)as follows

$$P(u_1) - P(u_2) = \prod_{s=0}^{e_d - 1} \frac{g(u_1 - 1^{s/e_d} u_2) g(u_1 + 1^{s/e_d} u_2)}{g^2(u_1) g_2(1^{s/e_d} u_2)} \quad (3.8)$$

For e^d=1 one has $\left(2E_d\right)^{\left(\frac{2^8}{d}\right)^{1/2}} \approx 1$ with precision $e^{-\pi\sqrt{d}}$ where E_d is a fundamental unit of N[$\sqrt{\ }$d]. Thus, modular units require at least three cycles. This corresponds to Kummer extensions $g_1^{g_2^{\dots g_n}}$ of generators g_i in (4.4) where n≥3 [49].

Modular units as modular congruences of elliptic units are chords and tangents of absolute quadrics Q_{abs} (1.2) and $Q_{abs}(r)$ (1.3) in close relationship to Poncelet closure. A continuous map of an interval I→I' into itself has a cycle of period m of a Poncelet polygon as a generator $1^{1/m}$ [g(u, L)].

Matrices (3.2) and (3.5) applied to reducible hyperelliptic theta functions (1.14) allow one- dimensional complex maps P^2, P^3, S^2 → C. Polygon points of P^3 correspond to star points of variable chords of the twisted cubic C_{tw}. The star generation theorem is based implicitly on branch points at θ''=ϑ(u'') and θ'''=ϑ(u''') which ramify into θ and θ', respectively. Four points formulate a variable cross ratio identity in §11 [9].

$$\lambda = \frac{\vartheta\left(u - u'''\right)\vartheta\left(u' - u''\right)}{\vartheta\left(u - u''\right)\vartheta\left(u' - u'''\right)} \quad (3.9)$$

corresponding to E_λ. At star points θ and θ' hyperelliptic variables branch as follows θ≺ θ'', θ''' and θ≺ θ'', θ''' supporting implicit bifurcation. Even ℘(qsω)=℘(-q_sω)=℘(ZL -q_sω) =℘(T_qω) satisfy a q_s-dependent complex tent map T_q

$$T_q\left(z\right) = \left\{ qz : 0 \geqslant argz \geqslant \pi / 2; q\left(\frac{Z}{q} - z\right) : \pi \geqslant argz \geqslant \pi / 2 \right\} \quad (3.10)$$

Reduction implies a proportionality coefficient (1.21) between hyperelliptic or elliptic functions and their first derivatives having rational values. Then second order derivatives of σ(u)σ(u') e.g.

$$\alpha_c^4 \left(\alpha D\right)^2 \sigma\left(u\right)\sigma\left(u'\right) \quad (3.11)$$

depend on operator (1.21). This treatment implying a limit u→u' (9) is crucial for understanding quantum statistics as an expansion into pseudo-random values A_q (4.2) of the Weierstrass or Jacobi zeta function. Poncelet involution $i^2(A_q)$=1 (see Appendix) then yields an expansion of the universal covering u into correlation functions δA_q $\delta A_q'$.

A Riemann surface X_2 as two layers X_1 of an universal covering δX has periods ω and ω' in C/(Z + ωZ) with ω' = Mω.

Arbitrary maps $u_{si}^{\circ k}$ generate a geometric string. Modular congruences mod α^{F_k} for involution matrix α and k^{th} Fermat number F_k have roots of unity α=2 and α=3 [52,53]. An expansion of u in terms of α yields the geometric zeta function of a fractal string

$$\zeta_{\mathcal{L}}^{(N)}\left(z\right) = \sum_{k \in N} w_k l_k^z \quad (3.12)$$

which differs from by congruence modulo F_k [54].

For multiplicity $w_k=2^k$ of string $l_k\big|_{N\to\infty} = 3^{-k-1}$ the Cantor string zeta function

$$\zeta_{CS}\left(z\right) = \sum_{k \in N} 2^k 3^{-(k+1)z} \quad (3.13)$$

is of interest in relation to MNT [54].

α^{F_k} replaced by $\det^{F_k}\alpha$ and e. g. α= $\gamma_\wp$∘ $\gamma_\wp$' summed over l_k= det$\gamma_\wp$=k yields the Riemann zeta function

$$\zeta\left(z\right) = \sum_{k \in N} k^{-z} \quad (3.14)$$

Next a topological partition function ζ_{top} is defined as follows

$$\zeta_{top}\left(h_{top}\right) = \sum_{k \in N} \zeta\left(h_{top}\left(\gamma_\wp^{\circ 2^k}\right)\right) \quad (3.15)$$

where ∘2^k denotes a 2^k fold map $\gamma_\wp$ ∘…∘$\gamma_\wp$ and ζ_{top} is regarded as congruent modulo $M_{2^{2^{K_i}}}$ which allows to define a DFT- $2^{2^{K_i}}$. The topological entropy h_{top} measures the complexity of bifurcations as a growth rate of orbits $\gamma_\wp$ [54].

$$h_{top}\left(\gamma_\wp^{\circ 2^k}\right) = 2^k h_{top}\left(\gamma_\wp\right) \quad (3.16)$$

with Hermite transformations $\gamma_\wp$.

It is assumed that h_{top}= U+ λ($\gamma_\wp$)depends additively on the Lyapunov exponent λ of the one- dimensional Hermite map $\gamma_\wp$ where

$$\lambda\left(\gamma_\wp\right) = \left(M_{2^k} + 1\right)^{-1} \sum_{i=0}^{M_{2^k}} \ln \gamma_\wp'\left(\wp_i\right) \quad (3.17)$$

The addition theorem on Eλ can be formulated in terms of invariant cubic polynomials Φ=f^3 -g_2f-g_3 which leads to

$$\gamma_\wp' \rightarrow \frac{\phi(t)}{(t-f)^2} \quad (3.18)$$

Matrix (3.18) describes SE(1) orbits (see Appendix) on complex plane.

The 2-power map in (3.16) allows a binary representation of 2^k= $(1+1)^k$. Now a binary decomposition of ln(n) for module $M_{2^{2^{K_i}}}$ leads to

$$\zeta_{top} = \sum_{k,n} e^{-2^k \ln(n) h_{top}\left(\gamma_\wp\right)} = \sum_{k,\mu} \prod_l e^{-(1+1)^k n_l 1^{2^{-l}} h_{top}\left(\gamma_\wp\right)} \quad (3.19)$$

This allows a fast multiplication of large numbers $\left(1+1\right)^k n_l 1^{2^{-l}}$ resulting e. g. in a complex constant c_k and a complex topological entropy h_{top}=Reh_{top} +i(S_{dyn} -U_{dyn}) where the dynamical entropy S_{dyn} and the energy U_{dyn} arises from SE(1) orbits. One gets the qualitative result

$$\zeta_{top}\left(h_t\right) = \sum e^{-c\left(U_{dyn} - S_{dyn}\right)} \quad (3.20)$$

as an envelope to the oscillatory part Reh_{top}. This expression is close to the definition of the partition function in quantum statistics.

In the limit k→∞ Weierstrass ℘- functions $z = \wp(u_{si}^{\circ k})$ contain a pole of second order at $\tilde{u}_{si}^{\circ k} \in C$ with $\tilde{u}_{si}^{\circ k} \notin ML$ where combinatorial dynamics creates a continuous map of an $u_{si}^{\circ k}$ interval I=[0,1]into itself.

The universal covering space δX are genus one surfaces and layers with both vanishing Euler- Poincaré characteristic χ(δX) as well first Chern class c_1(δX).

4. Coordinate variables and spin

Theta functions, $u^{\circ k}$ iterates as well γ- invariant hyperelliptic differential invariants (1.8) and (1.11) do not explain a ½ spin property. The addition theorem (3.1) for elliptic functions (3.8) is originally formulated in terms of ζ(u,L) and ℘(u,L) [55,24]. The flex point condition (1.5) transforms into a matrix of order 3x3. Integrating ζ(u_i) and ℘(u_i) with constants u_0, ζ_0, $\wp_0$, matrices of order 3 and 4

$$\begin{vmatrix} 1 & \zeta_1' & \wp_1' \\ 1 & \zeta_2' & \wp_2' \\ 1 & \zeta_3' & \wp_3' \end{vmatrix} = \begin{vmatrix} u_1 - u_0 & \zeta_1 - \zeta_0 & \wp_1 - \wp_0 \\ u_2 - u_0 & \zeta_2 - \zeta_0 & \wp_2 - \wp_0 \\ u_3 - u_0 & \zeta_3 - \zeta_0 & \wp_3 - \wp_0 \end{vmatrix} = \begin{vmatrix} M_0 & u_0 & \zeta_0 & \wp_0 \\ M_1 & u_1 & \zeta_1 & \wp_1 \\ M_2 & u_2 & \zeta_2 & \wp_2 \\ M_3 & u_3 & \zeta_3 & \wp_3 \end{vmatrix} = \det A\left(M=1,u,\zeta,\wp\right) = 0$$

(4.1)

resemble hyperelliptic (2.2) and (3.4). Index i= (0,1,2,3) denotes k-1, k-2, k-3, k-4 in u^{0i}. Parameters M_k=1 in (4.1) consider CM endomorphism on δX.

Unimodular collineations $\begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix}$ in (1.5) and (4.1) are modular invariant allowing Euclidean divisibility in Q[√d] [56]. Then C∞ differentiability holds for M∈ Q[√d] and k ∈ C for theta functions ϑ(u, ML) for abelian number fields M ∈ Q[√d] at CM.

The differential operator (1.21) acting on the product (3.8) of elliptic units (3.6) replaces $\partial / \partial u$ by $A_{q_S} = \zeta\left(q_S\omega\right) - \omega\eta$

$$\left(D_0, D_1\right) = \left(A_{q_0} - A_{q_0'}, A_{q_1} - A_{q_1'}\right) \tag{4.2}$$

At each iterate $u_{si}^{\circ k}$ i=(1,2,3,4) a new rational quadruple {q_s } s = (0, 0', 1, 1') has been detected as pseudo- random numbers[57].

Rational values {q_s} can be transformed into fractional characteristics g, h ∈Q^2. Fractional characteristics are equivalent to higher order theta functions. Finite g, h ∈Q^2 yield a translate

$$\vartheta\begin{bmatrix} g \\ h \end{bmatrix}\left(u,\omega\right) = 1^{\left(½\omega g^2 + g(u+h)\right)}\vartheta\left(u + \omega g + h, \omega\right) \tag{4.3}$$

of Jacobi (Riemann) theta functions

$$\vartheta\left(u,\omega\right) = \vartheta\begin{bmatrix} 0 \\ 0 \end{bmatrix}\left(u,\omega\right) = \sum_{m\in Z} 1^{½\omega m^2 + um}$$

A translate (4.3) is an one-dimensional Bloch state $1^{gh} \simeq e^{ikx}$ where g, h ≃ wave vector k and position x [58].

Modular units are inside cyclotomic units of modular groups Γ(N). To describe a close-to-an-integer-value at h_d=1 in K[∂]

$$\left(2g_d\right)^{\left(\frac{2^8}{d}\right)^{1/2}} \approx 1 \tag{4.4}$$

at minimum two generators μ and μ' or two- dimensional DFT or CM(2) are needed for unit g_d` .

In the following power- 2 generators of μ and μ' appear in both agM and BCF algorithms.

<u>Claim 4 on Spin property</u>

Iterated universal coverings $u_{si}^{\circ k}$ yield fluctuations of rational qs values us=q_sω on δX for s = {0, 1, 0', 1'}. The index s is called a spin component if k gets complex and a rotatory component $\varepsilon_{ijkl}u_iu_ju_ku_l$ with Levi- Civita tensor ε_{ijkl} gets complex under MNT(K_3) for CM(3) in SP in the limit K_1, K_2, K_3 →∞ where MNT(K_1) ∘CRT(K_2)∘FNT(K_2)∘MNT(K_3) are highly composite maps.

<u>Proof:</u> Bezout's identity contains $\binom{k_2-1}{2}$ pairwise coprime FNT(K) divisors for product (1.6). If the greatest common divisor of â, â+ ∈N is

$$gcd\left(\hat{a}_l, \hat{a}_l^+\right) = \hat{a}_l^+\hat{a}_l + \hat{a}_l\hat{a}_l^+ = 1 \tag{4.5}$$

CRT implies that uniformization parameter us as well Weierstrass or Jacobi zeta functions ζ are bilinear in creation â and annihilation operator $\hat{a}^+$

$$u_S = \sum_l u_{sl}n_{sl} = \sum_l u_{sl}\hat{a}_{sl}^+\hat{a}_{sl} \tag{4.6}$$

$$\zeta = \sum_l \zeta_l n_l = \sum_l \zeta_l \hat{a}_l^+\hat{a}_l \tag{4.7}$$

At CRT(K_2) the idempotent $n_l = \delta_{ll}$' modulo $M_{2^{K_2}}$ appears.

Two sources of complex q_s appear in SP:

1. Subsequent MNT(K_3) with $2^{2^{K_3}}$ complex roots of unity yield â, $\hat{a}^+$ ∈C where $\hat{a}_l^+\hat{a}_l + \hat{a}_l\hat{a}_l^+ = 1 mod M_{2^N}$

2. K_2-1 imaginary units i(FNT(K)) = $2^{2^{K-1}}$ exist where FNT (K+iπ)= -MNT(K)

For K_i<8 uniformization u_s=q_sω is ambiguous up to ½Û L constituting a group of order 256. For K <5 imaginary units i(FNT(K)) depend on prime integers FNT(K) with 2 and 3 as roots of unity. The first four Fermat numbers F_e= F_0, ..., F_4 are prime with generator 3^{F_l-1} [53].

Then FNT(K) followed by a MNT(K_3 →∞) yields a complexification of operators â and â+ in Bezout's identity. MNT(K_2) generators are Nth roots of unity.

It holds $1^{2^{-K}}$ ≃i(FNT(K))·i(FNT(K)).

SP for cubic irrationalities requires a power tower of generators $g_1^{\cdot\cdot\cdot g_l}$ with at least two cyclotomic units allowing a fast algorithm for 2-power bases. In the limit k→∞ the iteration parameter k gets complex.

The addition theorem (4.1) contains homogeneous u,ζ,℘ invariant with respect to CM multiplier u_s → M_{us}. The homogeneity condition holds also for P- functions in (3.7) and (3.8) as a product of generators and cyclotomic units.

Ambiguity of universal covering δX vanishes for $N_{edd} = 2^{2^8}$ th roots of unity where N_{edd} is the Eddington number which is identified with the number of fermions within the universe [59]. SP means averaging over three maps $\gamma_\wp$∘ $\gamma_\wp$'∘ $\gamma_\wp$'', i.e. steps k, k+1, k+2, k+3 where a partial map $\gamma_\wp$ is equivalent to multiplication by a modular unit as source for period-2 doubling.

In classical fluid dynamics uniformization u is time and theta function ϑ(u,L) is velocity potential. Four views u_s on δX correspond to $\binom{4}{3}$ combinations to project 4 roots onto 3 roots reducing a quartic polynomial to a cubic polynomial by means of a linear substitution. The Weierstrass ℘-function ℘(u, L) projects from torus T to foliations of a sphere S^2 . The paper investigates foliations of a sphere S^2 indexed by coordinates. Subsequent branched coverings δX generate a quadtree structure of branched-points $\{r_i\}$ on four spheres S^2 in analogy to the Einstein cross.

Addition (4.1) on E_λ of points k, k+1, k+2 holds equally for u, ζ and ℘.

Fermions are SP visions of fast highly composite power- 2 algorithms, where a quadtree index by s or k cannot be resolved The minimal rank of u is the minimal rank of $\Gamma_{k\to k+1}$ giving u_s. The independence of addition theorem (3.8) on multiplier M in ML transmits to an independence on Mu, M∈C . Rational quantities depend on modular units in L and their complex conjugate in $\bar{\Lambda}$.

Claim 5 on Zeros of AX

Zeroes of AX= 0 for rational X are realized if symbolic Hessians $(ab)^2 a_c b_c$ of cubic polynomials in (1.8) and (1.11) vanish, e. g. by trivial variables $a=\sigma_x c$.

Proof: For AX=0 both sides of hyperelliptic addition (3.1) vanish for a point X in space. Invariant- theoretic equations (1.8) and (1.11) contain Hessians $H=(ab)^2 a_c b_c$ of a cubic polynomial . Modular invariance implies invariant equations

$$(\theta\phi)^2 (b\theta)(b\phi) = 0,\ a_\theta^3 = 0,\ a_\phi^3 = 0,\ D_c^4 = 0$$

$$(ab)^2 a_c b_c = \pm a_c^2 (aD) \qquad (4.8)$$

The first and fourth equation a Hessian of a cubic polynomial. A vanishing Hessian reduces the polynomial to a pure cubic number [23]. The cubic Hessian is proportional to a discriminant of a quadratic polynomial where $(ab)^2 = (a_o b_1 - a_1 b_0)^2 = \Delta$ .This invariant theoretic expression is non-symbolic if ∃ a_i, b_i ∈ Q.

Generating pseudo- random values $D\in Q^2$ in (4.2) by solving Diophantine equations the condition (aD) = $\sqrt{\Delta}$ is discussed with pseudo- random $a\in Q^2$.

The symmetrized rationalized

$(aD) = \sum_{s_1,s_2=0,1,0',1'} c_{s_1,s_2} A_{q s_1} \tilde{A}_{q s_2}$ with css'=∓1 can be cast into the form

$\left(D_0\tilde{D}_1 - D_1\tilde{D}_0\right) = \sum_{i,i=1,\dots,4}\left[A_{q_i}, \tilde{A}_{q_i}\right]$. Diophantine solutions of reduced hyperelliptic variables yield $(aD)^2\in Z$ in (1.11). Furtheron, the modular invariant discriminant $(aD)^2$ reflects a CM endomorphism on surfaces K and W where (aD) ∈ MK[∂].

The discriminant of a matrix Ω of rank two

$$\left(D\tilde{D}\right)^2 = \det \hat{q}_{s_0 s_0', s_1 s_1'}(\Omega) \qquad (4.9)$$

is a determinant of a $2^2 \cdot 2^2$- matrix $\hat{q}(\Omega) = 1\otimes\Omega - \Omega\otimes 1$ defined in terms of $\phi_s = \varphi_s \otimes \varphi_{s'} - \varphi_s \otimes \varphi_{s'}$ with Ω eigenfunctions φs, s, s'=0, 0' ,1, 1' [60]. With hyperelliptic sigma functions σ(u) an invariant-theoretic expression for zeta functions would be $\sigma^{-2}(u)$ (aD)σ(u)σ(u') which is defined by A_q- fluctuations. A modular invariant differential $Z_0 du_0^+ Z_1 du_1$ for hyperelliptic zeta functions entering (1.15) supposes $u_0=\gamma u_1$ or $q_o\omega=\gamma q_1\omega$ or $A_{q_0} = \gamma A_{q_1}$ which requires to simulate a pseudo- random matrix γ. The integral ∫ζdv with ζ= ln'σ as entering the S- matrix (1.19) contains the correlation function $[A_{q_i}, \tilde{A}_{q_i}]$ as a square root of (4.9). Thus $\phi_s \simeq [A_{q_i}, \tilde{A}_{q_i}]$ are modular invariant, implying that (1.15) on δX reduces to elliptic sigma functions.

A decomposition of (1.21) in terms $C_M(3)$

$$\phi_s = \sum_{ll'} \phi_{sll'} \psi_l^+ \psi_{l'} \qquad (4.10)$$

bilinear with respect to â and its conjugate $\hat{a}^+$ reads

$$\psi_l = \sum_{l'} c_{ll'} \hat{a}_{l'},\ \psi_l^+ = \sum_{l} \bar{c}_{ll'} \hat{a}_{l'}^+ \qquad (4.11)$$

An eigen decomposition q(Ω)= Q diag (Γ) Q with orthogonal matrix Q=[ϕ_s] formed from Γ(Ω) eigenvectors Φ[ψ+, ψ] and diagonal matrix of eigenvalues diag(Γ(Ω)) yields [61-64]

$$(D\tilde{D}) = \det^{1/2} \hat{q}_{s_0 s_0', s_1 s_1'}(\Omega) = \det(\hat{q}_{s_0 s_0', s_1 s_1'}(\Omega))^{1/2} = \det Q^{1/2} \wedge^{1/2} Q^{-1/2}$$

(4.12)

The discriminant (4.9) Δ depends on a fourth power of pseudo-random values A_q of elliptic zeta function. Next the determinant $\det\left(\hat{q}_{s_0 s_0', s_1 s_1'}(\Omega)\right)^{1/2}$ is expanded into 2x2 minors

$$\sum_{ss'=(1,2,3,4)} \Gamma^{\phi}_{ss'} \Gamma^{\phi}_{c(ss')} = \Gamma^{\phi}_{12}\Gamma^{\phi}_{34} + \Gamma^{\phi}_{13}\Gamma^{\phi}_{24} + \Gamma^{\phi}_{14}\Gamma^{\phi}_{32} = \sqrt{\Delta} \qquad (4.13)$$

where c(ss') is a complementary set of $\binom{4}{2}$ cofactor indices s, s'.

The minors Γ^ϕ of square root (4.12) written in terms of eigenfunctions (4.10) are

$$\Gamma^{\phi} \to \bar{\psi}_{s_1}\bar{\psi}_{s_2}\Gamma_{s_1 s_2 s_1' s_2'}\psi_{s_1'}\psi_{s_2'} \qquad (4.14)$$

An algorithm to solve (4.14) inserted into (4.13) starts with monogenic fields iteratively replacing (4.11) e. g. by a sum over cyclotomic units μ, μ'. In accordance with claim 2 μ, μ' of 2- power cyclotomic fields is well known [14]. The Jacobi zeta function ζ generalizes vector potentials [3]. Here ζ fluctuations A_q dominate (4.13) with explicit dependence on A_q of degree deg Δ= 4, deg D = 1 , deg $\Gamma^\Phi \leq 2$.

The invariant $(aD)^2$ supports a controversial theory of invariants applied to the problems of chemical valences where e. g. the invariant $(oh)^2$ symbolizes water.

The modular invariance of Weierstrass sigma functions and of modular units as generators of cyclic fields implies fluctuations of the Weber- Schlaefli invariants f (ω). The Lyapunov exponent for (3.18)

$$\lambda(f) = N^{-1}\sum[\ln i\pi + \ln(t^3 - g_2 t - g_3) - 2\ln(t-f)] \qquad (4.15)$$

depends on mean t-f-fluctuations.[65,66].

CONCLUSION

The elliptic addition theorem has a pseudo-random component used in cryptography. Here a pseudo-periodic component is investigated as a recurrent random walk in one and two dimensions.

$C_M(1)$ cycles of u, ζ, ℘ on universal covering δX are related to SE(1) dynamics. The partition function (3.20) is in relation to coordinates and Euclidean number. Cycles $C_M(2)$ have a longitudinal and a transverse component and are called bosons. Cycles $C_M(3)$ have a longitudinal, transverse and rotatory component and are called spinor fields.

Processes (1.8), (1.11) and (4.13) describe an SP information current I whose equilibrium state is a sum of positive and negative rational values with I=0 ∈ Z. Fermions are bilinear idempotent n_l with congruences a, ā in $C_M(3)$.

Generators μ, μ are $2^{2^{K-1}}$ and $2^{2^{K-1}}$ roots of unity as one dimensional representations $D^{½,0} \otimes D^{0,½}$ of the rotation group where $D^{j,j'} = D^{j} \otimes D^{j'}$ is a direct product.
Binary invariants arise from Aronhold processes $\delta i \leftarrow \frac{\partial i}{\partial a_i} b_i$ which are invariant with respect to fractional substitutions γ ∈ SL(2, Z). δi is non-symbolic if a_i, b_i∈ Q

A bilinear representation $V = \sum_{ll'} v_{ll'} \bar{\psi}_l \psi_{l'}$ where $\psi_l = \sum_k c_l \hat{a}_{lk}$ of u, ζ, ℘ , ϑ or σ is a fast 2-power decomposition modulo M_{2K_1} , M_{2K_2} and M_{2K_3} .
As a result the Bethe- Salpeter equation (4.13) and Feynman diagrams obey a CM endomorphism End(Eλ/K) ≅{M∈C: ML⊂ L} for discriminant (4.9) with M ∈ Q [√ d] and L∈ K[∂].

Summation in (4.10) and (4.11) is over theta function characteristics (4.3) of one dimensional complex maps of ϑ(u) having cycles with an one-dimensional wave vector. The relevant function $\wp_i^{(2)}, \wp_i^{(3)}$ on sheets s = 0,1,0',1' of δX projects according to (1.20) from torus T to sphere S^2. The product of hyperelliptic σ(u) σ(u') as a product of four elliptic theta (4.3) on δX leads to a 4 dimensional generator exp(ik_i x_i) with k_i x_i (i=1,2,3,4) which corresponds to microstates where $\sum_l \rightarrow \sum_{k_1 k_2 k_3 k_4}$.
Bloch states arrange themselves as reducible g=3 theta functions leading to hyperelliptic Weierstrass functions $\wp_i^{(2)}, \wp_i^{(3)}$
Minkowski spacetime (x, ct) ∈ $Q^{3,1}$ with rational x , complex parameter c and complex continuous time t ⇐ t+ iτ is realized in the limit k →∞ where k ∈ C gets complex.
The addition theorem (4.1) for u, ζ and ℘ depends on bilinear compositions (4.11). The determinant (4.1) vanishes if $u^{\circ k}$, $u^{\circ k+1}$ and $u^{\circ k+2}$, $\zeta^{\circ k}$, $\zeta^{\circ k+1}$ and $\zeta^{\circ k+2}$ and $\wp^{\circ k}$, $\wp^{\circ k+1}$ and $\wp^{\circ k+2}$ undergo unimodular collineations.

Due to ζ=ln' σ(u)≃q_sωln σ(u) the N th iterative of (4.1) yields a N th order determinant in terms of log g(q_sω,L) where q_s=(r, s)∈Q^2 which is equivalent with the regulator R of the system of modular units (3.6) with ML. For modular groups Γ(N) the regulator R is given by a circulant matrix of elliptic units. The Slater determinant implies the presence of a power tower of generators g(u,L) as a product of sigma functions is equivalent to the regulator R of units in Q [√ d] K [∂].
The square root √Δ is a limit of an expansion of quantum statistical scattering processes in terms of cyclotomic approximations of vertices Γ and states ψ.

Appendix 1 Poncelet theorem for quadrics in space

The Poncelet theorem for quadrics in P^2 and P^3 and the addition theorem for elliptic functions (4.1) are equivalent. As a result indices s=0, 1, 0', 1' correspond to a quadruple k-1, k-2, k-3, k-4 in (3.3) on δX. The involution matrix α_{ss}' depends on parameters a_k, b_k and c_k in (3.3) algebraically.

Rational quadrics (1.9) map θ ∈ P^1 to a point on the twisted cubic C_{tw}. A Weddle surface W(X) = $\sum x_i Q_i(X)$ as a pencil of 4 quadrics Q_i(X) ∈ P^3 is projected onto δX as 6 pencil of 2 quadrics Q_i(X) ∈ $\mathbb{P}^3$. A pencil of two quadrics Q_i(X) ∈ P^3 splits into four conics in $\mathbb{P}^2$. The Poncelet closure theorem states that infinity of rational solutions exists if a closed n- polygon in P^2 and a 4 polytope in P^3 is formed. A partial line (P, T) of a closed 4-polytope of dimension s=0, 1, 0', 1' consists of points P and tangents T=dP/du (P, T)≃($x^{\circ k}$, $x^{\circ k+1}$,$x^{\circ k+2}$) satisfying (4.1). The closure condition is a periodic pair of involutions i_x(P,T) and i_x'(P,T)

$$\left(i_x\left(P,T\right)\circ i_x'\left(P,T\right)\right)^n = 1 \qquad (5.1)$$

with i_x(P, T)= (P, T'), i_x(P, T)=(P' ,T) and E→E . Since $i_x^2 = 1$ and $i_x'^2 = 1$ involutions i_x, i_x' induce involutions i_u(s, s') and i_u'(s, s') on universal covering δX ∈ C4 on sheets us of δX according to

$$i_u\left(s\right) = \sum_{s=1}^{4} \alpha_{ss'} u_{s'} + \omega_s \qquad (5.2)$$

with the involution condition $i_u^2(s) = 1$ modulo ML.
The closure condition is an identity map on δX
$i_u(s_1)\circ i_u(s_2)\circ i_u(s_3)\circ i_u(s_4)=1$
or

$$i_u^2\left(s\right) = \sum_{s',s''=1,\dots,4} \alpha_{ss'}\alpha_{s's''} u_{s''} + \sum_{s'=1,\dots,4} \left(\alpha_{ss'} + \delta_{ss'}\right)\omega_{s'} \qquad (5.3)$$

The branched covering δX consists of sheets s=0, 1 and s=0', 1'. Functions u, ζ and ℘ in addition theorem (4.1) with index quadruple (0 ,1 ,2 ,3)= (0, k, k+1, k+2) on δX yield$\binom{4}{3}$triples which arrange K=2 groups with κ=1,2.

Two involutions i_x and i_u yield four matrices α(κ), α(K +κ) which form a group G_{32} of order 32. The elements of G_{32} are $\left(-1\right)^{b_0} \alpha^{b_1}\left(\kappa_1\right)\alpha^{b_2}\left(\kappa_2\right)\alpha^{b_3}\left(\kappa_3\right)\alpha^{b_4}\left(\kappa_4\right)$ with b_i=0,1 where
α(κ') α(κ)=-α(κ) α(κ')
$\alpha^2(\kappa)=1$
(α(κ)-+1)τ=0 mod ML.
On a definite sheet of a quadruple {0, k, k+1, k+2} one has α = -1 ∈ R. A closure of the 4-polytope takes place if

$$\sum_{s=1,\dots,4} \omega_s = QL \qquad (5.4)$$

leading to u= q_sω∈ML (r,s)∈Q^2 which emphasizes that the Poncelet theorem and the addition theorem are closely related to modular units.
Each iteration k generates a Kronecker product A(M, u, ζ, ℘)→A(M, u, ζ, ℘)⊗ A(M, u, ζ, ℘) of matrix A(M, u, ζ, ℘) in (4.1) and of involution matrices α⊗α → α.

Performing k steps α→ α^2 one has $\alpha^{2^{2^k}}$.
Complex parameter a, a+ in (4.5) are related to α- matrices by

$$\alpha_{ss'}\left(\kappa\right) = a_{ss'}\left(\kappa\right) + a_{ss'}^{+}\left(\kappa\right)$$
$$\alpha_{ss'}\left(K+\kappa\right) = -i\left(a_{ss'}\left(\kappa\right) - a_{ss'}^{+}\left(\kappa\right)\right) \qquad (5.5)$$

where

$$a(\kappa)a^{+}(\kappa') + a^{+}(\kappa')a(\kappa) = \delta_{kk'},$$
$$a\left(\kappa\right)a\left(\kappa'\right) + a\left(\kappa'\right)a\left(\kappa\right) = 0,$$
$$a^{+}\left(\kappa\right)a^{+}\left(\kappa'\right) + a^{+}\left(\kappa'\right)a^{+}\left(\kappa\right) = 0$$

Subsequent k generate imaginary units i(FNT(k)) via CRT, MNT and FNT decompositions and a complexification. The relevant group of order 32 for complex a, a+ consists of Gamma matrices and Dirac matrices Γ_μ . Individually u, ζ and ℘ reflect the symmetries of G_{32}. As shown in §18 and §44 of 32 places x, X of det K(x)=0 , det W(X)=0 and 32 tangents (2.3) constitute group of the order 32.

Appendix 2 Transition from robot dynamics to chaotic dynamics

CF, BCF and CCF matrices M(a), M(a_1, a_2) and M(a_1, a_2, a_3) with n=1,2,3 in (3.5) and (3.2) are given by

$$M\left(a\right)=t_n+\begin{pmatrix}0_n & 0\\ 0_n & a^T\end{pmatrix} \qquad (6.1)$$

with a cyclic matrix of order n+1 $\hat{t}_{n+1}$, $\hat{t}_{n+1}^{n+1}=1, \hat{t}_{n+1}\hat{t}_{n+1}^{T}=1$ $0_{n,m}$ a zero matrix with n rows and m columns. One has

$$t_{n+1}^{T}M\left(a\right)=\begin{pmatrix}1 & a^T\\ 0_{1,} & 1\end{pmatrix}=e^{S(a)} \qquad (6.2)$$

which contains one and two- dimensional collineations if {a_i=0 ∧ a_j=0} or {a_i=0} ∀i, j=(1,2,3). Even for chaotic collineations invariances exist. Firstly (1.5) is invariant with respect to Hermite transformations (2.4). Secondly a nilpotent N_0 exists for ∀M(a) which can be written as in terms of Dirac matrices Γ_μ. The idempotents P_0, $P_\mp$ are related to projection operators $\frac{1}{2}\left(1\pm\gamma_5\right)$ for four- component bases.

Variables x= (x, 1) transform under action of the special Euclidian group SE(n) for n- cycles shifted via $\hat{t}_{n+1}^{T}$.

For different rotation matrices R_k one has

$$G\left(R,a\right)=\begin{pmatrix}R & x\\ 0 & 1\end{pmatrix}=e^{S}$$

where $S=\begin{pmatrix}R & x\\ 0 & 0\end{pmatrix}$ and R^2 <0 and

$$\prod_{k=1}^{4}\begin{pmatrix}\Omega_k & x\\ 0 & 1\end{pmatrix}=\begin{pmatrix}\Omega_1\Omega_2\Omega_3\Omega_4 & \Omega_1\Omega_2\left(\Omega_3+1\right)x+\left(\Omega_1+1\right)x\\ 0 & 1\end{pmatrix}$$

Then M(a) corresponds to a quadruple of imaginary rotations R_k for position vector $x\rightarrow t_4^{T}x$ and ambiguous rotations R_k. The CCF matrix allows a fast with R a rotation matrix with R^4=1

The equation for the matrix S yields the following idempotent P_0, P_+, P_- and the nilpotent N_0

$$P_0=1-\frac{S^2}{\Omega^2}$$

$$P_\pm=\frac{1}{2\Omega^3}S^2\left(\Omega\mp S\right)$$

and

$$N_0=\frac{1}{\Omega^2}S^2(\Omega-S^2)$$

as 4x4 matrices which are independent on a definite vector x. In dependence on P_0, P_+, P_- and the nilpotent N_0 the matrix S reads

$$e^{S}=P_0+N_0+e^{-\Omega}P_{+}+e^{\Omega}P_{-} \qquad (6.3)$$

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