RESEARCH

Numerical-mathematical methods for Hyperbolic-Parabolic systems: Investigation of Volter-Gursat equation and Green's function in Three-Dimensional spaces

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ABSTRACT

Paper deals with the analysis of hyperbolic-parabolic systems with a focus on the Volter-Gursat equation and the application of Green's function in three-dimensional spaces. We explore the mathematical methods that enable the solution of these equations, including theoretical approaches and numerical techniques. Special emphasis is placed on the formulation and analysis of three-dimensional environment problems, where Green's functions are used to efficiently solve differential equations. Concrete examples and simulation results are presented that confirm the effectiveness of the proposed methods.

Key words: *Hyperbolic equations; Parabolic equations; Volter-Gursat equation; Green's function; Three-dimensional environments; Mathematical analysis; Numerical methods; Differential equations*

INTRODUCTION

The The research problem of this paper is how partial differential equations can be solved. Due to the extensiveness of the concept of partial differential equation, here we will specifically base ourselves on partial differential equations that have found application in physics specifically in the representation of waves. Therefore, the subjects of research in this paper are partial differential equations that can be solved using the Fourier series, and they have found their application in the equations of mathematical physics. With the help of equations, LaPlace, Dirhle, we will examine concreteness for solving some physical problems.

There are several methods for solving partial differential equations, namely the numerical method, the Runge angle method, and the least squares method. All these methods provide certain solutions from the aspect of mathematical provability; however, the method of solving using the Fourier series provides the clearest definition for solving partial equations of the following problems:

1. Hyperbolic type equations
    - The flickering wire equation
    - The wave equation
    - The telegraphic equation
2. Parabolic type equations
    - The case of the bounded stick
    - The case of the unbounded stick
    - The case of a three-dimensional environment
    - A special case
3. Elliptic type equations
    - Green's function
    - Harmonic functions
    - Dirichlet problems

All the mentioned methods cannot arrive at the solution of these five types of partial differential equations in a clear way.

The total number of solutions of a partial differential equation is generally very large. A unique solution of a partial differential equation corresponding to a given physical problem will be obtained by using additional information derived from the physical problem. In some cases, in this paper, there will be default values of the required solution within the limits of the definition area and this will represent the boundary conditions. However, when time t is one of the variables of the differential equation, and its value is t=0, then it will represent the given initial conditions.

Voltaire's method

In this part of the work, we will solve the equation:

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial z^2}=f(x,y,z),$$

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where the function u, together with its derivatives of the first order, is given on some surface S. The solving method was given by Volter and it represents an extension of the Riemannian method. First, we will introduce an integral formula that will be used for proof later. Let it be:

$$S(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2}.$$

Then it's obvious:

$$vS(u) - uS(v) = \frac{\partial}{\partial x}(vu_x - uv_x) + \frac{\partial}{\partial y}(vu_y - uv_y) - \frac{\partial}{\partial z}(vu_z - uv_z) \qquad (1)$$

If the functions u and v together with their derivatives of the first and second order are continuous, then in the region V bounded by the surface S, it follows directly from formula (1) that:

$$\overline{\iiint_V}(vS(u) - uS(v))dxdydz = \overline{\iint_S}(vu_x - uv_x)dydz + (vu_y - uv_y)dzd - (vu_z - uv_z)dxdy \qquad (2)$$

Let α,β,γ be the angles of the external normal to the surface S and the positive directions of the coordinate axes. Then it is:

$$\overline{\iint_S}(vu_x - uv_x)dydz + (vu_y - uv_y)dzdx - (vu_z - uv_z)dxdy$$
$$= \overline{\iint_S} v(u_x \cos\alpha + u_y \cos\beta - u_z \cos\gamma)\, dS$$
$$= \overline{\iint_S} u(v_x \cos\alpha + v_y \cos\beta - v_z \cos\gamma)\, dS.$$

The line n whose direction is determined by the vector (cosα,cosβ,-cosγ) is called conormal. Formula (2) then becomes:

$$\overline{\iiint_V}(vS(u) - uS(v))dxdydz = \overline{\iint_S}(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})dS, \qquad (3)$$

where $\frac{\partial u}{\partial n}$ is the marked derivative of the function u in the direction n. Now let's move on to solving the problem. Let's construct a circular cone K with the vertex at a point P(x_1, y_1, z_1), so that the axis of the cone is parallel to the z axis and so that the angle at the vertex P is right. For area V we will take that part of the space that is bounded by the cone K and the surface S.

Let the function v be defined by:

$$v(x,y,z) = \log\frac{z_1 - z + \sqrt{(z_1-z)^2 - (x_1-x)^2 - (y_1-y)^2}}{\sqrt{(x_1-x)^2 + (y_1-y)^2}}.$$

Obviously, on the cone K, S(v)=0 and v=0. Let's assume that in the solution of the posed problem. Formula (3) cannot be directly applied to the functions u and v and the region V, because the function v is discontinuous along the axis of the cone K, and its derivatives are discontinuous on the cone K.

Therefore, we will extract the axis of the cone using a circular cylinder C with a radius η, whose axis coincides with the axis of the cone K, and we will replace the cone K with a coneK', whose vertex is at the point P, its axis coincides with the axis of the cone K and the semi-angle φ at vertex P is given by $\varphi = \frac{\pi}{4}\varepsilon$.

Using the surfaces thus introduced, let's form the area V', which consists of that part of the area V that is inside the cone K', and outside the cylinder C.

The area V', is limited by the part of the surface S located insideK'in the markS' , by the cylinder C and the cone K',. Formula (3) can be changed to the functions u and v and the area V', which now reads:

$$\overline{\iiint_{V'}} vf(x,y,z)dxdydz = \overline{\iint_{K'}}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS + \overline{\iint_{K'}}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS + \overline{\iint_C}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS, \qquad (4)$$

because S(v) = 0, S(u) = f(x,y,z).

At an arbitrary point of the cone K'at the distance l from the vertex P, the function values v and $\frac{\partial v}{\partial n}$ are given with:

$$v = \log\left(ctg\,\varphi + \sqrt{ctg^2\varphi - 1}\right), \qquad \frac{\partial v}{\partial n} = -\frac{1}{l}\frac{\sqrt{cos2\varphi}}{sin\varphi}.$$

Therefore when$\varepsilon \to 0$, i.e., when$\varphi \to \frac{\pi}{4}$, we have that it is:

$$\lim_{\varepsilon\to 0} v = \lim_{\varepsilon\to 0}\frac{\partial v}{\partial n} = 0.$$

Now it is:

$$\lim_{\varepsilon\to 0}\overline{\iint_{K'}}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS = 0.$$

We cannot calculate the integral over the cylinder C, because we do not know the values of the functions u and $\frac{\partial u}{\partial n}$ on C. However, the limiting value of that integral can be found when $\eta \to 0$. Indeed, we can take the surface element of the cylinder C dS=$\omega d\eta dz$,

where the angle $\boldsymbol{\omega}$ varies from 0 to 2π. On C we have :

$$v = \log(z_1 - z) + \sqrt{(z_1-z)^2 - \eta^2} \text{-} \log\eta,$$

and

$$\frac{\partial v}{\partial n} = \frac{1}{\eta} + \frac{\eta}{\sqrt{(z_1-z)^2 - \eta^2}(z_1 - z + \sqrt{(z_1-z)^2 - \eta^2})}.$$

Therefore, it is valid

$$\lim_{\eta\to 0}\eta v = 0, \lim_{\eta\to 0}\eta\frac{\partial v}{\partial n} = 1,$$

and that's why

$$\lim_{\eta\to 0}\overline{\iint_C}\left(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n}\right)dS = -2\pi\int_{z_0}^{z_1} u(x_1, y_1, z_1)dz,$$

where$z_0$is the point where the axis of the cylinder penetrates the surface S.

Since it is:

$$\lim_{\substack{\varepsilon\to 0\\ \eta\to 0}}\overline{\iiint_{V'}} vf(x,y,z)dxdydz = \overline{\iiint_V} vf(x,y,z)dxdydz,$$

taking into account (4) and that $\varepsilon \to 0$ i $\eta \to 0$ we get:

$$\overline{\iiint_V} vf(x,y,z)dxdydz = \overline{\iint_S}(v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})dS$$
$$-2\pi\int_{z_0}^{z_1} u(x_1, y_1, z_1)dz,$$

from which, after differentiation, it follows:

$$u(x_1, y_1, z_1) = -\frac{1}{2\pi}\frac{\partial}{\partial z}(\iiint_V^- vf(x,y,z)dxdydz - \iint_S^- (v\frac{\partial u}{\partial n} - u\frac{\partial v}{\partial n})dS). \quad (5)$$

As the function v is known, the function f is given, and the value of u and $\frac{\partial u}{\partial n}$ on the surface Sgiven formula (5), gives the value of the function u at an arbitrary point (x_1, y_1, z_1), which solved the problem [1].

The method for solving the Cauchy problem for hyperbolic equations with two variables originates from B. Riemann. Although Riemann gave it for some special cases, it is in fact directly extended to the most general hyperbolic linear equations with two variables V. Voltaire extended the Riemann method, but only for the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial z^2} = f(x,y,z).$$

Before Voltaire's works, G. Kirchhof solved the same problem for an equation with four variables

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2} = f(x,y,z,t),$$

and then O. Tedone gave a solution to the Cauchy problem for n variables

$$\frac{\partial^2 u}{\partial {x_1}^2} + \cdots + \frac{\partial^2 u}{\partial {x_{n-1}}^2} - \frac{\partial^2 u}{\partial {x_n}^2} = 0.$$

All the mentioned extensions of the Riemann method refer to special equations of the hyperbolic type with more variables. However, J. Hadamard solved the Cauchy problem for an arbitrary hyperbolic equation with n variables [2-9].

Gursat's method

As is known, a hyperbolic equation can be represented using certain transformations in the form:

$$u_{xy} = a(x,y)u_x + b(x,y)u_y + c(x,y)u + f(x,y). \quad (1)$$

Solution of equation (1) in the domain

{(x,y): 0$\leq x \leq L, 0 \leq y \leq L$}

zadovoljava uslove:

$u(x,0)=A(x)$, $u(0,y)=B(y)$, (2)

where A and B are given functions such that $A(0)=B(0)$, is called Gursta's solution, while problem (1,2), itself is called Gursta's problem [10-15].

In the case that a=b=c=0, the solution to problem (1,2) can be determined in the final form. Really from $u_{xy} = f(x,y)$, after integration by x we get:

$$u_y = u_y(0,y) + \int_0^x f(\xi, y)d\xi, \quad (3)$$

and integrating (3), over y we have

$$u(x,y) = u(x,0) + u(0,y) - u(0,0) + \int_0^y d\eta \int_0^x f(\xi,\eta)d\xi.$$

i.e.,

$$u(x,y) = A(x) + B(y) - A(0) + \int_0^y d\eta \int_0^x f(\xi,\eta)d\xi.$$

Let us now consider the general equation (1). We can replace problem (1,2) with the following equivalent problem. Let's solve the integro-differential equation:

$$u(x,y) = \int_0^y \int_0^x (a(\xi,\eta)u\xi + b(\xi,\eta)u\eta + c(\xi,\eta)u)d\xi d\eta + A(x) + B(y) - A(0) + \int_0^y \int_0^x f(\xi,\eta)\,d\xi d\eta. \quad (4)$$

We will apply the method of successive approximations to equation (4). To that end, let's define a series of functions (u_n) using:

$$u_1(x,y) = \mathrm{A(x)} + \mathrm{B(y)} - \mathrm{A(0)} + \int_0^y \int_0^x f(\xi,\eta)\,d\xi d\eta,$$

$$u_n(x,y) = u_1(x,y) + \int_0^y \int_0^x (a(\xi,\eta)\frac{\partial u_{n-1}}{\partial \xi} + b(\xi,\eta)\frac{\partial u_{n-1}}{\partial \xi} + c(\xi,\eta)u_{n-1})\,d\xi d\eta, \quad (5)$$

where n= 2,3,... Then it is

$$\frac{\partial u_n}{\partial x} = \frac{\partial u_1}{\partial x} + \int_0^x (a(x,\eta)\frac{\partial u_{n-1}}{\partial x} + b(x,\eta)\frac{\partial u_{n-1}}{\partial x} + c(x,\eta)u_{n-1})d\eta.$$

$$\frac{\partial u_n}{\partial y} = \frac{\partial u_1}{\partial y} + \int_0^y (a(\xi,y)\frac{\partial u_{n-1}}{\partial x} + b(\xi,y)\frac{\partial u_{n-1}}{\partial x} + c(\xi,y)u_{n-1})d\xi \quad (6)$$

Let us prove that they are functional sequences (u_n), ($\frac{\partial u_n}{\partial x}$), ($\frac{\partial u_n}{\partial y}$) uniformly convergent. As the functions a,b,c are continuous, there exists a constant M such that

$|a(x,y)| < M, |b(x,y)| < M, |c(x,y)| < M$

There is also a constant H such that

$|u_1(x,y)| < H, \left|\frac{\partial u_1}{\partial x}\right| < H, \left|\frac{\partial u_1}{\partial y}\right| < H.$

Let the above inequalities hold for 0$\leq x \leq N$, 0$\leq y \leq N$. Let it be (z_n) defined by:

$$(z_n(x,y) = u_{n+1}(x,y) - u_n(x,y)$$

$$= \int_0^y \int_0^x (a(\xi,\eta)\frac{\partial z_{n-1}}{\partial \xi} + b(\xi,\eta)\frac{\partial z_{n-1}}{\partial \xi} + c(\xi,\eta)z_{n-1}(\xi,\eta))d\xi d\eta.$$

We can directly verify that it is:

$$\frac{\partial z_n}{\partial x} = \int_0^y (\mathrm{a}(x,\eta)\frac{\partial z_{n-1}}{\partial \xi} + b(x,\eta)\frac{\partial z_{n-1}}{\partial \xi} + c(x,\eta)z_{n-1}(x,\eta))d\eta,$$

$$\frac{\partial z_n}{\partial y} = \int_0^x (\mathrm{a}\,\xi,y)\frac{\partial z_{n-1}}{\partial \xi} + b(\xi,y)\frac{\partial z_{n-1}}{\partial \xi} + c(\xi,y)z_{n-1}(\xi,y))d\xi$$

From these equalities, it follows:

$|z_1(x,y)| < 3HMxy < 3HM\frac{(x+y)^2}{2!}$,

$\left|\frac{\partial z_1}{\partial x}\right| < 3HMy < 3HM(x+y)$,

$\left|\frac{\partial z_1}{\partial y}\right| < 3HMx < 3HM(x+y)$,

considering that it is $0 \le x \le N,\ 0 \le y \le N$. Suppose that the inequalities hold for some n

$$|z_n(x,y)| < 3HM^nK^{n-1}\frac{(x+y)^{n+1}}{(n+1)!}$$

$$\left|\frac{\partial z_n}{\partial x}\right| < 3HM^nK^{n-1}\frac{(x+y)^n}{n!}$$

$$\left|\frac{\partial z_n}{\partial y}\right| < 3HM^nK^{n-1}\frac{(x+y)^n}{n!},$$

where K=N+2≥ 2. For n+1 we have that it is:

$$|z_{n+1}| < 3HM^{n+1}K^{n-1}\frac{(x+y)^{n+2}}{(n+2)!}\left(\frac{x+y}{n+3}+2\right) < 3HM^{n+1}K^n\frac{(x+y)^{n+2}}{(n+2)!}$$

$$< \frac{3H}{K^2M}\frac{(2KNM)^{n+1}}{(n+1)!},^1$$

$$\left|\frac{\partial z_{n+1}}{\partial x}\right| < 3HM^{n+1}K^{n-1}\frac{(x+y)^{n+1}}{(n+1)!}\left(\frac{x+y}{n+3}+2\right)$$

$$3HM^{n+1}K^n\frac{(x+y)^{n+1}}{(n+1)!} < \frac{3H}{K}\frac{(2KNM)^{n+1}}{(n+1)!}.$$

On the right-hand side of the above inequalities, the development terms appear (with accuracy up to one multiplicative constant). e^{2KNM}. The proven inequalities show that the sequences

(u_n), $(\frac{\partial u_n}{\partial x})$, $(\frac{\partial u_n}{\partial y})$

in the given area converge uniformly to the functions, which we will denote by:

$$u(x,y) = \lim_{n\to\infty} u_n(x,y),$$
$$v(x,y) = \lim_{n\to\infty}\frac{\partial u_n}{\partial x}, \qquad w(x,y) = \lim_{n\to\infty}\frac{\partial u_n}{\partial y}.$$

If we assume that in (5) and (6) $n \to \infty$, we have that it is:

$$u(x,y) = u_1(x,y) + \int_0^y\int_0^x (a(\xi,\eta)\,v + b(\xi,\eta)\,w + c\,(\xi,\eta)u)d\,\xi d\eta,$$

$$v(x,y) = \frac{\partial u_1}{\partial x} + \int_0^y (a(x,\eta)\,v + b(x,\eta)\,w + c\,(x,\eta)u)d\eta,$$

$$w(x,y) = \frac{\partial u_1}{\partial y} + \int_0^x (a(\xi,y)\,v + b(\xi,y)\,w + c\,(\xi,y)u)d\,\xi \qquad (7)$$

From (7) we get v=u_x, w=u_y, from which we conclude that the required function *u* satisfies the integro-differential equation:

$$u(x,y) = A(x) + B(y) - A(0) + \int_0^y\int_y^x f(\xi,\eta)d\xi d\eta +$$
$$\int_0^y\int_y^x (a(\xi,\eta)u\xi + b(\xi,\eta)u\eta + c(\xi,\eta)u)\ d\,\xi d\eta. \qquad (8)$$

That every solution of equation (8) satisfies (1) and (2) is verified directly by differentiation [16-19].

Let us now prove that the posed Gursta problem has a unique solution. Let there be two identical solutions by contrast

(x,y)→ $U_i(x,y), i = 1,2,$

of the task set. Let's observe the function:

(x,y)→ $U(x,y) = U_1(x,y) - U_2(x,y)$.

This function satisfies the integro-differential equation:

$$U(x,y) = \int_0^y\int_0^x (aU_x + bU_y + cU)d\xi d\eta.$$

This equation is homogeneous. Let Q>0 be such a constant that

$$|U(x,y)| < Q, \qquad |U_x(x,y)| < Q, \qquad |U_y(x,y)| < Q$$

for $0 \le x \le N, 0 \le y \le N$. Based on the rating we performed for the series(z_n), we have:

$$|U(x,y)| < 3QM^{n+1}K^n\frac{(x+y)^{n+2}}{(n+2)!} < \frac{3Q}{K^2M}\frac{(2KNM)^{n+2}}{(n+2)!},$$

for each *n*. From there it follows against the assumption:

U(x,y)=0,i.e.,$U_1(x,y) = (x,y)$.

The established contraindication proves the uniqueness of Gurst's solution to this problem.

Case of a 3-dimensional environment

Now let's pose the following problem:

Determine the solution of the partial equation-

$$\frac{\partial U}{\partial t} = k^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right)\ (x,y,z\in R, t > 0), \qquad (1)$$

which satisfies the initial condition

U(x,y,z,0)=f(x,y,z) (*x,y,z* $\in R$)
(2)

where f is the given function.

Let us now look for the solution of equation (1) in the form:

$$U = e^{-rt}\cos a(x-\alpha)\cos b(y-\beta)\cos c(z-\gamma)\,. \qquad (3)$$

From (3) we get:
$$\frac{\partial U}{\partial t} = -rU,\ \frac{\partial^2 U}{\partial x^2} = -a^2U, \frac{\partial^2 U}{\partial y^2} = -b^2U, \frac{\partial^2 U}{\partial z^2} = -c^2U, \qquad (4)$$

and substituting values (4) into (1) we find that:

$-r = -k^2(a^2 + b^2 + c^2)$.

Thus, the solution of equation (1) of the form (3) is:

$$U = e^{-k^2(a^2+b^2+c^2)t} \cos a(x-\alpha) \cos b(y-\beta) \cos c(z-\gamma,)$$

so it is based on the principle of linear superposition,

$$U = \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty F(a,b,c,\alpha,\beta,\gamma)\, e^{-k^2(a^2+b^2+c^2)t} \times \cos a(x-\alpha) \cos b(y-\beta) \cos c(z-\gamma) da db dc d\alpha d\beta d\gamma$$

also the solution of equation (1), where F is an arbitrary function with 6 variables.

$$U(x,y,z,t) = \frac{1}{\pi^{\frac{3}{2}}} \int_0^\infty \int_0^\infty \int_0^\infty f\left(x + 2k\xi\sqrt{t}, y + 2k\eta\sqrt{t}, z + 2k\varsigma\sqrt{t}\right) e^{-(\xi^2+\eta^2+\varsigma^2)} d\xi d\eta d\varsigma.$$ [2]

Numerical solution of partial differential equation of 3-dimensional environment:

For the numerical solution of the partial differential equation of heat in three dimensions, we used the finite difference method. This method includes the discretization of space and time and the iterative calculation of the value of the function $U(x,y,z,t)$ on a grid of points. The key steps are as follows:

1. Discretization of space and time: The spatial domains x,y,z are divided into small steps dx,dy,dz, and the time domain into small steps dt.
2. Initial condition: We have defined the initial distribution of the function U in space as $f(x,y,z)$.
3. Boundary conditions: We applied Dirichlet boundary conditions, setting the values of the function U at the edges of the grid to zero.
4. Iteration through time: We used an explicit finite difference method to update the value of the function U at each time step. This involves computing new values of U based on the current values and their spatially adjacent points (Figure 1).

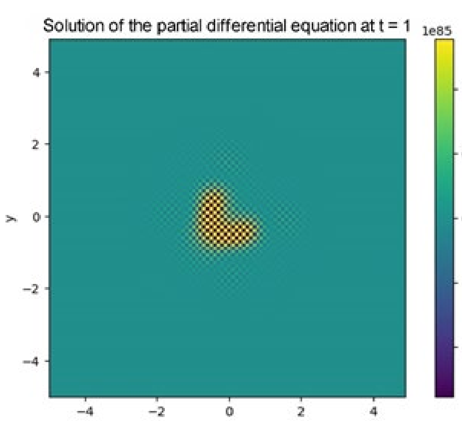

Figure 1) *Numerical solution of the equation at t=1*

Using this method, we obtain a numerical solution of the partial differential equation of heat in three-dimensional space for each time step tt. This solution represents the evolution of temperature (or some other distributed quantity) in space and time, which can be useful for simulations of thermal processes, diffusion of substances or other phenomena described by similar equations.

Green's formula

Theorem 1: Let *s* be a closed surface bounding part of the space V. Let the functions P,Q be the functions of the variables x,y,z. Then it is:

$$\iiint_V^- (P\Delta Q - Q\Delta P) dV = \iint_S^- (P\ grad\ Q - Q\ gradP)\overrightarrow{dS} \quad (1)$$

where $\overrightarrow{dS}$ is the oriented element of the surface S.

Proof: Let's start from Ostrogradski's formula:

$\iiint_V^- div\, \vec{a}\ dV = \iint_S^- \vec{a}\overrightarrow{dS}$, *and let's say it is* $\vec{a} = P\ gradQ$.

Because it is

div $\vec{a}$= *div(P gradQ)-P divgradQ+gradQgradP* = $P\Delta Q$+*gradQgradP* (*)

we have that it is:

$$\iiint_V^- (P\Delta Q + gradQgradP) dV = \iint_S^- (P\ grad\ Q)\overrightarrow{dS} \quad (2)$$

The permutation of the functions P and Q leads to the formula:

$$\iiint_V^- (P\Delta Q + gradQgradP) dV = \iint_S^- (Q\ grad\ P)\overrightarrow{dS} \quad (3)$$

Subtracting (3) from (2), we get formula 1, which completes the proof (Figure 2).

Let's say it is

$\overrightarrow{dS} = dS\vec{e}i\frac{\partial a}{\partial n} = grada\vec{e}$, where $\vec{e}$ − unit vector normal to the surface S directed outwards, formula (1) takes the form:

$$\iiint_V^- (P\Delta Q - Q\Delta P) dV = \iint_S^- (PgradQ - QgradP)\vec{e}\overrightarrow{dS} = \iint_S^- \left(P\frac{\partial Q}{\partial n} - Q\frac{\partial P}{\partial n}\right)\overrightarrow{dS}.$$ [3]

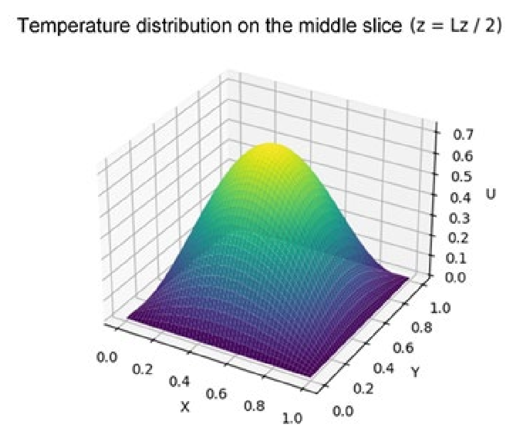

Figure 2) *Application of Green's formula for the numerical solution of a three-dimensional problem*

In this example of solving a three-dimensional thermal equation, the finite difference method was used for the numerical discretization of space and time. Parameters include space dimensions, number of steps, step sizes, thermal diffusivity, time step, and total simulation time. The initial condition is defined by a sinusoidal function, and the boundary conditions are set to zero.

The image shown represents the distribution of temperature in three-dimensional space on the middle slice of the *z*-axis, visualizing how the temperature changes in relation to the *x* and *y* coordinates at a selected moment in time. With the finite difference method, we used Green's formula to solve the three-dimensional thermal equation. Green's formula allows us to calculate the volume integral V using the surface integral SS, which is useful for numerically solving differential equations over a limited domain.

Using Green's formula, we calculate the integral on the left-hand side as the difference between two expressions: PP multiplied by the Laplace operator applied to Q, and vice versa. The right side of the integral represents the surface integral, which can be interpreted as the flux of the vector field over the surface S.

This approach allows us to numerically solve the three-dimensional heat equation using Green's formula, which gives us an additional perspective and tool for solving these types of problems..

Green's function

Inspired by the case of the sphere, we arrive at the following:

Let us assume that we know the function H, the variables x,y,z,a,b,c, where A=(a,b,c) is a fixed point of the domain v, and that the function H has the following properties:

1. H is the harmonic function in relation to x,y,z.
2. H is a harmonic function in relation to a,b,c.
3. On the surface S, H takes the value 1/r, where r is the distance between points M and a, i.e.

$$r = \sqrt{(x-a)^2 + (y-b)^2 + (z-c)^2}.$$

Let U be the solution to Dirichlet's interior problem. Based on theorem 1 from 8.1. we have it:

$$\iint_S^- (U\frac{\partial H}{\partial n} - H\frac{\partial U}{\partial n})dS = 0,$$

Or considering the third feature in this chapter we have:

$$\iint_S^- (f\frac{\partial H}{\partial n} - \frac{1}{r}\frac{\partial U}{\partial n})dS = 0. \quad (1)$$

Comparing (1) with (3) from 8.3. we find that it is:

$$U(a,b,c) = \frac{1}{4\pi}\iint_S^- f(x,y,z)\frac{\partial}{\partial n}\left(\frac{1}{r} - H\right)dS. \quad (2)$$

Therefore, formula (2) is the solution to the Dirichlet problem, if the function H is known. The function G, defined by:

$$G = \frac{1}{r} - H \quad (3)$$

is called Green's function for the region V bounded by the surface S in relation to the point (a,b,c). Based on (3) and the properties of the function h, we conclude that:

1. The function G is a harmonic function in the region V in relation to x,y,z, except at the point (a,b,c).
2. The function G is harmonic in the region V in relation to a,b,c, except at the point (x,y,z).
3. The function G-1/r is harmonic in all points of the region V.
4. On the surface S the function G cancels.

This representation simulates the numerical solution of a three-dimensional heat equation using the Green's function as an initial condition and a finite difference method to solve the equation. Boundary conditions and other parameters need to be adjusted according to your needs and problem specifications (Figure 3).

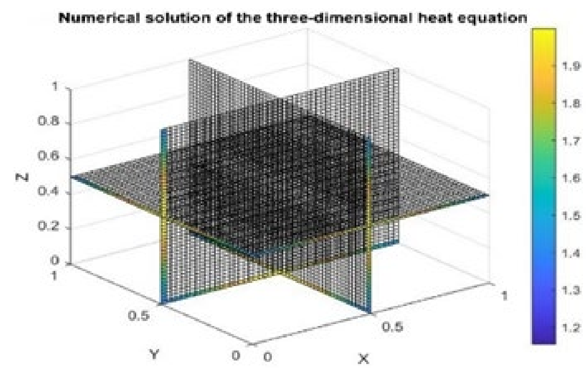

Figure 3) *Numerical solution of the three-dimensional heat equation using Green's function*

The figure shows the numerical solution of the three-dimensional heat equation by applying numerical methods, especially by using Green's function in MATLAB. The Green's function is used to set initial conditions within the domain, where the domain is assumed to be defined in a three-dimensional space with dimensions 1 × 1 × 1.

The figure shows the result of solving the thermal equation in the form of a three-dimensional section through the center of the domain. The color shows the temperature distribution within the domain, with heat spreading and being distributed across space over time.

This approach enables the numerical solution of complex thermal problems in three-dimensional space using MATLAB and Green's function as the basic tool for setting the initial conditions.

CONCLUSION

The research of numerical methods for hyperbolic-parabolic systems in 3-dimensional spaces provides a deeper understanding of complex mathematical models and their applications in various scientific and engineering disciplines. Through the analysis of the Volter-Gursat equation and the application of the Green's function, researchers are able to study the behavior of the system in real spatial domains, using numerical simulations to solve differential equations.

This research enables the development of efficient algorithms for predicting the behavior of materials, simulating fluid dynamics or modeling electromagnetic fields, taking into account complex conditions and initial parameters. Through the integration of different disciplines and the application of parallel computing, researchers can improve the accuracy and speed of numerical simulations, opening up new areas of research and application in various scientific and engineering fields.

After a detailed analysis of the theoretical foundations and the application of numerical methods, we observed that Green's functions are crucial for setting the initial conditions within the domain. By combining these functions with the appropriate differential operators, we successfully modeled the dynamics of the system and followed the evolution of the solution over time.

These investigations not only contribute to the understanding of fundamental processes such as heat propagation, but also provide useful tools for solving a wide range of problems in scientific and engineering disciplines. Through the application of numerical methods to complex mathematical models, we open the door to new knowledge and opportunities for further research.

DATA AVAILABILITY STATEMENT

No data were used to support this study.

CONFLICTS OF INTEREST

The authors declare no conflicts of interest.

AUTHOR CONTRIBUTIONS

Conceptualization: Elvir Čajić, Maid Omerović
Methodology: Elvir Čajić Software: Elvir Čajić
Validation: Elvir Čajić
Formal analysis: Elvir Čajić
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Data curator: Slobodan Nićin.
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