MINI REVIEW

On KC spaces and Wallman compactification

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ABSTRACT

This article deals with a characterization of spaces with a KC Wallman compactification. A description of a such space is given. We also establish necessary and sufficient conditions on particular topological spaces to have their Wallman compactification KC-spaces.

Keywords: Clinical; Biochemical; Therapeutic; Goats

INTRODUCTION

A topological space is called a KC-space if every compact set of it is closed. Since Hausdorff spaces are KC-spaces and every KC-space is a T_1-space, it is natural to see the KC property as a separation axiom.

In Hajek proved that if A is a compact set of the Wallman compactification of a T_3-space X then A ∩ X is a closed set of X.

The characterization of spaces such that their Wallman compactification satisfy a given separation axiom was a subject of several recent research papers (see for example) [1-6]. Hence it is natural to wonder what conditions should check a topological space to have its Wallman compactification a KC-space

The first section of this paper contains some remarks and properties of the Wallman compactification of a T_1-space.

The second section deals with a characterization of spaces such that their Wallman compactification are KC-spaces.

In the third section (resp. fourth section) we establish a necessary and sufficient conditions on a w-space (resp. space with finite Wallman remainder) in order to have its Wallman compactification a KC-space.

In section five we give some remarks about spaces such that their one-point compactification is KC-space.

Some remarks about the Wallman compactification First, recall that the points of the Wallman compactification wX of a T_1-space X are the closed ultrafilters on X [7,8]. The base for the open sets of a topology on wX is {U* | U is an open set of X} with U* {F ∈ wX | F ⊆ U for some F in F}, and {D* | D is a closed set of X} with D*={F ∈ wX | D ∈ F} is a base for closed sets of the topology on the Wallman compactification wX.

In authors called an open cover U of a topological space X a good covering (g-covering, for short) of X if it has a finite subcover, and U is called a bad covering (b-covering, for short) of X if it is not a g-covering [2].

The following remarks are frequently useful.

Remarks 1.1: Let X be a non-compact T_1space.

If U is a g-covering of X and F ∈ wX\X then there exists U ∈ U such that F ∈ U*. In fact, since U is a g-covering of X, there exists a finite subcollection U' of U such that X=∪ (U: U ∈ U'). Hence wX=∪ (U*: U ∈ U'). Thus, there exists U ∈ U' such that F ∈ U∗.

Let F ∈ wX\X. The collection U of open sets U of X such that U* ⊆ wX\{F} is a b-covering of X and wX\∪ (U*: U ∈ U)={F}.

We need the following definition to describe a particular class of b-covering of a T_1-space.

Definition 1.2: Let X be a non-compact T_1-space. A b-covering U of X is called a 1-b-covering of X if for each two open sets O_1 and O_2 of X such that U ∪ {O_1, O_2} is a g-covering of X, either U ∪ {O_1} or U ∪ {O_2} is a g-covering of X.

Proposition 1.3: Let X be a non-compact T_1-space. An open cover U of X is a 1-b-covering of X if and only if there exists F ∈ wX\X such that wX\∪ (U *: U ∈ U)={F}.

Proof: Necessary condition: That wX\∪ (U*: U ∈ U) 6= ∅ follows immediately from the fact that U is a 1-b-covering of X. Suppose that there exist two distinct elements F, G ∈ wX\X such that {F, G} ⊆ wX\∪ (U*: U ∈ U). Let O_1 be an open set of X such that F ∈ O^*_1 and G ∈/ O^*_1. So there exists a closed set F ∈ F such that F ⊆ O_1 and F ∈ G. Set O_2=X\F. Since O_1 ∪ O_2=X, {O_1, O_2} is a g-covering of X. Hence U ∪ {O_1, O_2} is a g-covering of X, contradicting the fact that U of X is a 1-b-covering of X, since either U ∪ {O_1} and U ∪ {O_2} are a b-covering of X. Thus wX\∪ (U*: U ∈ U) is a singleton.

Sufficient condition: Let O_1 and O_2 be two open sets of X such that U ∪ {O_1, O_2} is a g-covering of X. Then F ∈ O^*_1 ∪ O^*_2; so that either U ∪ {O_1} or U ∪ {O_2} is a g-covering of X. Therefore, U of is a 1-b-covering of X.

Let X be a non-compact T_1-space and U be a 1-b-covering of X. The element F of wX\X such that {F}=wX\∪ (U*: U ∈ U), will be denoted by FU.

The following corollaries are immediate consequences of proposition 1.3 and remarks 1.1.

Corollary 1.4: Let X be a non-compact T1-space. For all F ∈ wX\X, there exists a 1-b-covering U of X such that F=FU.

Corollary 1.5: Let X be a non-compact T_1-space. For all subset K of wX\X, there exists an ordered pair (A, U) with A is a subset of X and U is a collection of 1-b-covering of X such that K=A ∪ {FU | U ∈ U}.

Remark 1.6: Let U and V be two 1-b-covering of a non-compact T_1-space X such that FU=FV. An open set O of X is such that U ∪ {O} is a g-covering of X if and only if V ∪ {O} is a g-covering of X.

Definition 1.7. Let X be a non-compact T_1-space. Two 1-b-covering U and V of X are said w-equivalent, and denoted U ∼w V if for each open set O of X, U ∪ {O} is a g-covering of X if and only if V ∪ {O} is a g-covering of X. Two non w-equivalent 1-b-covering U and V of X well be denoted by U ≁w V.

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Proposition 1.8: Let U and V be two 1-b-covering of a non-compact T_1-space X. Then U and V are w-equivalent if and only if wX\∪ (U*: U ∈ U)=wX \∪(V∗ : V ∈ V).

Wallman compactification and KC-spaces

Our goal in the present section is to give necessary and sufficient conditions on a T_1-space in order to have its Wallman compactification a KC-space. First, we need the following definition.

Definition 2.1: Let X be a T_1-space, A be a subset of X and U be a collection of 1-b-covering of X. The ordered pair (A, U) is said to be w-closed if the following properties hold.

A is a closed set of X.

For all x ∈ X\A, there exists an open set O of X such that x ∈ O and U ∪ {O} is a b-covering of X, for each U ∈ U.

If V is a 1-b-covering of X such that U ≁w V, for each U ∈ U, then there exists an open set O of X such that V ∪ {O} is a 1-g-covering V of X, O ∩ A=∅ and U ∪ {O} is a 1-b-covering of X, for each U ∈ U.

Proposition 2.2: Let X be a T_1-space. A subset K of wX is closed if and only if (K ∩ X, U) is w-closed with U is a collection of 1-b-covering of X such that K ∩ (wX\X)={FU | U ∈ U}.

Proof: Necessary condition.

Since K is a closed set of wX, K ∩ X is a closed set of X.

Let x ∈ X\(K ∩ X). Since K is a closed set of wX, there exists an open set O of X such that x ∈ O and O* ∩ K=∅. Hence FU ∈/O*, for each U ∈ U. Thus U ∪ {O} is a b-covering of X, for each U ∈ U.

Let V be a 1-b-covering V of X such that U≁w V, for each U ∈ U. Then FV ∈/K. Since K is a closed set of wX, there exists an open set O of X such that FV ∈ O* and O* ∩ K=∅. Hence O ∩ (K ∩ X)=∅ and FU ∈/O*, for each U ∈ U. Thus U ∪ {O} is a 1-b-covering of X, for each U ∈ U.

Therefore (K ∩ X, U) is w-closed. Sufficient condition. Let K be a subset of wX such that (K ∩ X, U) is w-closed with U is a collection of 1-b-covering of X such that K ∩ (wX\X)={FU | U ∈ U}.

Let x ∈ X\(K ∩ X). Since (K ∩ X, U) is w-closed, K ∩ X is a closed set of X, so there exists an open set O of X such that x ∈ O and O ∩ (K ∩ X)=∅, and there exists an open set O' of X such that x ∈ O' and U ∪ {O'} is a b-covering of X, for each U ∈ U. Hence O'* ∩ (K ∩ (wX\X))= ∅. Thus (O ∩ O')* is an open neighborhood of x such that (O ∩ O')* ∩ K=∅.

Let F ∈ (wX\X) ∩ (wX\K). Then there exists a b-covering V of X such that F=FV and U≁w V, for each U ∈ U. Since (K∩X, U) is w-closed, then there exists an open set O of X such that V ∪ {O} is a g-covering of X, O ∩ (K ∩ X)=∅ and U ∪ {O} is a 1-b-covering of X, for each U ∈ U. Hence FV ∈ O*, O* ∩ (K ∩ X)=∅ and FU ∈/O*, for each U ∈ U. Thus O is an open neighborhood of x such that O* ∩ K=∅. Therefore, K is a closed set of wX.

We need the following definitions:

Definition 2.3: Let X be a T_1-space, A be a subset of X and U be a collection of 1-b-covering of X.

RESULTS AND DISCUSSION

A collection O of open sets X is said to be a w-cover of the ordered pair (A, U) if A ⊆ ∪ (O:O ∈ O) and for each U ∈ U there exists O ∈ O such that U ∪ {O} is a g-covering of X.

The ordered pair (A, U) is said to be w-compact if for each w-cover O of (A, U) there exists a finite subcollection O0 of O such that O' is a w-cover of (A, U).

Proposition 2.4: Let X be a T_1-space. A subset K of wX is compact if an only if the ordered pair (K ∩ X, U) is w-compact with U is a collection of 1-b-covering such that K ∩ (wX\X)={FU | U ∈ U}.

Proof: Necessary condition. Let O be a w-cover of (K ∩X, U) with U is a collection of 1-b-covering such that K ∩ (wX\X)={FU | U ∈ U}. Then K ∩ X ⊆ ∪ (O: O ∈ O) and for each U ∈ U there exists O ∈ O such that U ∪ {O} is a g-covering of X, so that FU ∈ O*. Since for each F ∈ K ∩ (wX\X), there exists U ∈ U such that F=FU, K ⊆ ∪ (O*: O ∈ O). Then there exists a finite subset O0 of O such that K ⊆ ∪ (O*: O ∈ O), since K is compact. Hence K ∩ X ⊆ ∪ (O: O ∈ O') and for each U ∈ U there exists O ∈ O0 such that U ∪ {O} is a g-covering of X. Thus (K ∩ X, U) is w-compact.

Sufficient condition. Let K be a subset of X such that (K ∩ X, U) is w-compact with U is a collection of 1-b-covering such that K ∩ (wX\X)={FU | U ∈ U}, and O be a collection of open sets of X such that K ⊆ ∪ (O*: O ∈ O). Hence K ∩ X ⊆ ∪ (O: O ∈ O) and for each F ∈ (wX\K) ∩ (wX\X) there exists O ∈ O such that F ∈ O*, so that U ∪ {O} is a g-covering of X with U is a 1-b-covering of X such that FU=F. Thus for each 1-b-covering U of X such that FU=F, U ∪ {O} is a g-covering of X. Then O is a w-cover of the ordered pair (K ∩ X, U). Since (K ∩X, U) is w-compact, there exists a finite subset O0 of O such that O0 is a w-cover of (K ∩ X, U). So K ∩ X ⊆ ∪ (O: O ∈ O0) and for each U ∈ U there exists O ∈ O0 such that U ∪ {O} is a g-covering of X, it follows that K ⊆ ∪ (O*: O ∈ O0). Therefore, K is a compact set of wX.

Now, we are in position to give a characterization of spaces such that their Wallman compactification is a KC-space.

Proposition 2.5: Let X be a T_1-space. Then the following statements are equivalent:

wX is a KC-space.

For each subset A of X and each collection U of 1-b-covering of X such that (A, U) is w-compact, (A, U) is w-closed.

Proof: (1) =⇒ (2) Let A be a subset of X and U be a collection of 1-b-covering of X such that (A,U) is w-closed. Set K=A ∪ {FU ∈ wX\X | U ∈ U}. Since (A,U) is w-compact, K is compact, by Proposition 2.4. Hence K is closed. Thus (A, U) is w-closed by Proposition 2.2.

(2) =⇒ (1) Let K be a closed set of wX. By Proposition 2.4, (K ∩ X,U) is closed with U is a collection of 1-b-covering such that K∩ (wX\X)={FU | U ∈ U}. Then (K ∩ X,U) is w-compact. Hence (K ∩ X,U) is w-compact. Therefore, wX is a KC-space.

Case of w-spaces

The goal of the present section is to give a characterization of w-spaces such that their Wallman compactification are KC-spaces.

First, let us recall that in order to give a characterization of T_1-spaces such that there Wallman compactification is a Whyburn space, authors of [3] introduced the notion of 1-closed set and a class of w-spaces as follows:

A subset C of a topological space X is called a 1-closed set if every two non-compact closed sets F_1 and F_2 of C meets.

A T1-space X is said to be a w-space if for each collection C of non-compact closed sets of X with the FIP there exists a 1-closed set N of X such that C ∪ {N} has also the FIP.

Authors of proved that every closed ultrafilter of a w-spaces contains 1-closed set, and that spaces with finite Wallman remainder are w-spaces. We adopt notations of, two subsets A and B of a T1-space X are said to be of w-intersection nonempty, and we denote A ⊓ B ≠ ∅, if there exists a non-compact closed set N of X such that N ⊆ A ∩ B. It is immediate that if F ∈ wX\X, F ∈ F and O is an open set of X such that F ∈ O∗ then there exists a non-compact closed set F 0 of X such that F 0 ⊆ F and F 0 ⊆ O. Then F⊓O ≠∅.

The following proposition highlights the very close relationship between covering and 1-closed of w-spaces.

Proposition 3.1: Let X be a w-space and U be a b-covering of X. The following statements are equivalent:

U is a 1-b-covering of X.

There exists a 1-closed set F such that U ∪ {O} is a g-covering, for each open set O of X such that $O \sqcap F \neq \emptyset$.

Proof. (i) =⇒ (ii) Since U is a 1-b-covering of X, there exists a unique F ∈ wX\X such that wX\∪ (U* :U ∈ U)=F. Then there exists a 1-closed set F of X such that F ∈ F, since X is a w-space. Let O be an open set of X such that $O \sqcap F \neq \emptyset$. Hence F ∈ O*. Thus U ∪ {O} is a g-covering of X.

(ii) =⇒ (i) Let F be a 1-closed set of X such that U ∪ {O} is a g-covering of X, for each open set O of X such that $O \sqcap F \neq \emptyset$. Set F be the unique element of wX−X such that F ∈ F (that F is unique is immediate from [3]). Since $O \sqcap F \neq \emptyset$, F ∈ O*.

Suppose that there exists U ∈ U such that F ∈ U*. Since U is a b-covering of X, there exists G ∈ wX − X such that G ∈ wX\∪ (U∗: U ∈ U). Hence F ≠ G. Thus there exists an open set O of X such that F ∈ O* and G ∈/O*. Then $F \sqcap O \neq \emptyset$ and U ∪ {O} is a b-covering, contradicting hypothesis. So that F ∈/ U*, for all U ∈ U. Let O_1 and O_2 open sets of X such that U ∪ {O1, O2} is a g-covering of X. Then ∪ (U*: U ∈ U) ∪ {O^*_1, O^*_2}=X. Then either F ∈ O^*_1 or F ∈ O^*_2. Without loss of generality, we consider that F ∈ O^*_1. Hence O_1 u F 6= ∅. Thus U ∪ {O_1} is a g-covering of X. Therefore, U is a 1-b-covering of X.

Now, we are in position to give a characterization of w-spaces such that their Wallman compactification is a KC-space.

Proposition 3.2: Let X be a w-space. Then the following statements are equivalent:

wX is a KC-space.

If A is a subset of X and F is a collection of 1-closed sets of X satisfying the following properties.

If U is an open cover of A such that, for each F ∈ F, there exists U ∈ U such that $F \sqcap U \neq \emptyset$, then there exists a finite subcollection U' of U such that U' is a cover of A and, for each F ∈ F, there exists U ∈ U' such that $U \sqcap F \neq \emptyset$.

Then

For each x ∈ X\A there exists an open neighborhood O of x such that A ∩ O=∅ and $F \sqcap O \neq \emptyset$, for each F ∈ F.

For each 1-closed set H of X such that H ⊓ F=∅, for each F ∈ F, there exists a open set O of X such that H ⊓ O 6= ∅, A ∩ O=∅ and F u O=∅, for each F ∈ F.

Proof. (i) =⇒ (ii) Let A be a subset of X and F be a collection of 1-closed sets of X satisfying the following properties.

If U is an open cover of A such that, for each F ∈ F, there exists U ∈ U such that $F \sqcap U \neq \emptyset$, then there exists a finite subcollection U' of U such that U' is a cover of A and, for each F ∈ F, there exists U ∈ U' such that $U \sqcap F \neq \emptyset$.

Set K=A ∪ {F ∈ wX\X | ∃F ∈ F and F ∈ F}. Then K is a compact set of wX. In fact, let V be an open cover of K. Hence V is an open cover of A such that, for all F ∈ F, there exists V ∈ V such that $F \sqcap V \neq \emptyset$. Thus there exists a finite subcollection V' of V such that V' is a cover of A and, for each F ∈ F, there exists V ∈ V0 such that $V \sqcap F \neq \emptyset$. Then V' is a finite subcover of K, so K is compact.

Now, since wX is a KC-space, K is a closed set of wX. Let x ∈ X\A. Then x ∈ X\A. Hence there exists an open neighborhood O of x such that K ∩ O*=∅. Thus A ∩ O*=∅ and F ⊓ O=∅, for each F ∈ F.

Let H be a 1-closed set of X such that H ⊓ F=∅, for each F ∈ F. Set H be the element of wX\X such that H ∈ H. Then H ∉ K. Hence there exists an open set O of X such that H ∈ O* and O* ∩ K=∅. Thus H ⊓ O 6= ∅, A ∩ O=∅ and F ⊓ O=∅, for each F ∈ F.

(ii) =⇒ (i) Let K be a compact subset of wX. Set A=K ∩X and F a collection of 1-closed sets of X such that for each F ∈ K ∩ (wX\X) there exists F ∈ F and F ∈ F.

Let U be an open cover of A such that such that, for each F ∈ F, there exists U ∈ U such that $F \sqcap U \neq \emptyset$. Then {U* | U ∈ U} is an open cover of K.

Since K is compact, there exists a finite sub collection U' of U such that {U* | U ∈ U'} is an open cover of K. Hence U' is a cover of A and for each F ∈ K ∩ (wX\X) there exists U ∈ U' such that F ∈ U*. Thus for each F ∈ F, there exists U ∈ U_0 such that $U \sqcap F \neq \emptyset$.

Let x ∈ wX\K. We discuss two cases:

Case 1: x ∈ X\K, so x ∈ X\A. Then there exists an open neighborhood O of x such that A ∩ O=∅ and F ⊓ O=∅, for each F ∈ F. Hence O* ∩ K=∅.

Case 2: x ∈ (wX\X)\K, so there exists H ∈ wX\X such that x=H. Since X is a w-space there exists a 1-closed set such that H such that H ∈ H. Since H / ∈ K, H ⊓ F=∅, for each F ∈ F. Hence there exists a open set O of X such that $H \sqcap O \neq \emptyset$, A ∩ O=∅ and F u O=∅, for each F ∈ F. Thus O* is an open neighborhood of H such that O* ∩ K=∅.

Therefore, K is a closed set of X.

Case of spaces with finite Wallman remainder

The goal of this section is to give a necessary and sufficient conditions on a T_1-space X with a finite Wallman remainder (that is, wX\X is finite) in order to get its Wallman compactification a KC-space. First recall that, Kovar [5] has proved that a T_1-space X has a finite Wallman remainder if and only if there exists n ∈ N such that every family of non-compact pairwise disjoint closed sets of X contains at most n elements, and X has a collection of n pairwise disjoint non-compact closed sets.

The following remarks give relationships between a collection of non-compact pairwise disjoint closed sets and 1-b-covering of a T_1-space with a finite Wallman remainder. We use notations adopted in the first section.

Remarks 4.1. Let X be a T1-space such that Card (wX\X)=n and F be a collection of n non-compact pairwise disjoint closed sets of X. Then the following statements hold.

For each F ∈ wX\X, there exists a unique F ∈ F such that F ∈ F.

For each subset K of wX there exist a subset A of X and a subcollection F' of F such that K=A ∪ {F ∈ wX\X | ∃F ∈ F, and F ∈ F}.

An open cover U of X is a 1-b-covering of X if and only if there exists unique FU ∈ F such that {F}=wX\∪ (U* : U ∈ U) with F is the unique F ∈ wX\X such that FU ∈ F.

If U and V are two 1-b-covering of X such that FU=FU then FU=FV.

For each F ∈ F there exists a 1-b-covering U of X such that F=FU.

Let A ⊆ X, F 0 ⊆ F and U be collection of 1-b-covering of X such that F ∈ F 0 if and only if there exists U ∈ U with F=FU. Then the ordered pairs (A, F') and (A, U) describe the same subset of wX. Hence we can say that an open collection O is a w-cover of (A, F') if O is a w-cover of (A, U), (A, F') is w-compact if (A, U) is w-compact and (A, F') is w-closed if (A, U) is w-closed.

Proposition 4.2: Let X be a T1-space such that Card (wX\X)=n, F be a collection of n non-compact pairwise disjoint closed sets of X and O be a collection of open sets of X. Then for each A ⊆ X and F 0 ⊆ F, the following statements are equivalent.

(1) O is a w-cover of (A, F').

(2) A ⊆ ∪ (O: O ∈ O) and for each F ∈ F 0 there exists O ∈ O such that $O \sqcap F \neq \emptyset$.

Proof. (1) =⇒ (2) Since O is a w-cover of (A, F'), A ∪ {F ∈ wX\X | ∃F ∈ F 0 and F ∈ F} ⊆ ∪ (O*: O ∈ O). Hence A ⊆ ∪ (O: O ∈ O) and for each F ∈ wX\X such that there exists F ∈ F 0 and F ∈ F, there exists O ∈ O such that F ∈ O*. Thus there exists F' ∈ F such that F' ⊆ O. So $O \sqcap F \neq \emptyset$.

(2) =⇒ (1) Let O be a collection of open sets of X such that A ⊆ ∪ (O: O ∈ O) and for each F ∈ F' there exists O ∈ O such that $O \sqcap F \neq \emptyset$. Then for each F ∈ wX\X such that F ∈ F 0 and F ∈ F, there exists O ∈ O such that F ∈ O*; so that A ∪ {F ∈ wX\X | ∃F ∈ F 0 and F ∈ F} ⊆ ∪ (O*: O ∈ O). Therefore, O is a w-cover of (A, F').

Proposition 4.3: Let X be a T_1-space such that Card (wX\X)=n, F be a collection of n non-compact pairwise disjoint closed sets of X and O be a collection of open sets of X. Then for each A ⊆ X and F 0 ⊆ F, the following statements are equivalent.

(1) (A, F') is w-closed.

(2) A is a closed set of X and for each F ∈ F\F', there exists an open set O of X such that F ⊓ O ≠ ∅ and O ∩ A=∅.

Proof. (1) =⇒ (2) Since (A, F') is w-closed, A ∪ {F ∈ wX\X | ∃F ∈ F' and F ∈ F} is a closed set of wX. Hence A is a closed set of X and clwX (A)\X ⊆ {F ∈ wX\X | ∃F ∈ F 0 and F ∈ F}. Thus for each F ∈ wX\X such that there exists F ∈ F with F ∈ F\F', F ∈/ clwX(A). Then there exists an open set O of X such that F ∈ O* and O ∩ A=∅. So F ⊓ O ≠ ∅ and O ∩ A=∅.

(2) =⇒ (1) Let A be a closed set of X and F 0 ⊆ F such that for each F ∈ F \F', there exists an open set O of X such that F ⊓ O ≠ ∅ and O ∩ A=∅. Then for F ∈ wX\X such that there exists F ∈ F\F' and F ∈ F then F ∈/ clwX (A). It turns out that {F ∈ wX \X | ∃F ∈ F\F' and F ∈ F} ⊆ wX \clwX(A). Now, since A is a closed set of X, A∪ {F ∈ wX \X | ∃F ∈ F' and F ∈ F}=clwX (A)∪G with G ⊆ wX \X. That G is a closed set of wX is immediate from the fact that wX \X is finite. So clwX (A) ∪ G is a closed set of wX. Therefore (A, F') is w-closed.

Now, we are in position to give a characterization of spaces with finite Wallman remainder such that their wallman compactification is a KC-space.

Proposition 4.4: Let X be a T_1-space such that Card (wX\X)=n and F be a collection of n non-compact pairwise disjoint closed sets of X. Then the following statements are equivalent.

(1) wX is a KC-space.

(2) For each F 0 ⊆ F and A ⊆ X such that (A, F) is w-compact, (A, F) is w closed.

Proposition 4.5: Let X be a T1-space such that Card (wX\X)=n. Then the following statements are equivalent.

wX is a KC-space.

If A is a subset of X such that A ∩ (X\O) is compact for each open neighborhood O of ∪ (F: F ∈ F) with F is a collection of n disjoint non-compact closed sets of X, then A is closed.

Proof: (i) =⇒ (ii) Let A be a subset of X such that A ∩ (X\O) is compact for each open neighborhood O of ∪ (F: F ∈ F) with F is a collection of n disjoint non-compact closed sets of X. Set K=A ∪ (wX\X) and let U be a collection of open sets of X such that K ⊆ ∪ (U*: U ∈ U). Since for all F ∈ wX\X there exists U ∈ U such that F ∈ U*, there exists a collection of n disjoint non-compact closed sets G of X such that for each G ∈ G there exists UG ∈ U such that G ⊆ UG. Set O=∪ (UG: G ∈ G). Hence A ∩ (X \O) is compact. Since A ∩ (X\O) ⊆ ∪ (U: U ∈ U), there exists a finite sub collection U' of U such that A ∩ (X\O) ⊆ ∪ (U: U ∈ U'). Then A ⊆ ∪ (U: U ∈ U') ∪ (∪ (UG: G ∈ G)), so K is a compact of X. Since wX is a KC-space, K is a closed set of wX. Therefore, A is a closed set of X.

(ii) =⇒ (i) Let K be a compact set of wX. Set A=K ∩ X and let O be an open neighborhood of ∪ (F: F ∈ F) with F is a collection of n disjoint non-compact closed sets of X. Then wX\X ⊆ O*. Let V be a collection of open sets of X such that A ∩ (X\O) ⊆ ∪ (V: V ∈ V). Hence K ⊆ O* ∪ (∪ (V: V ∈ V)). Since K is a compact set of wX, there exists a finite subcollection V' of V such that K ⊆ O* ∪ (∪(V : V ∈ V0)). Thus A ∩ (X\O) ⊆ ∪(V : V ∈ V_0), so A is compact. Then A is closed set of X and K is a closed set of wX.

Space such that its one-point compactification is a KC-space

It is immediate that if Card (wX\X)=1 then wX coincide with the one-point compactification. The following result is immediate consequence of Proposition 4.5.

Corollary 5.1. Let X be a T_1-space such that each two non-compact closed sets meet. Then the following statements are equivalent.

The one-point compactification of X is a KC-space.

If A is a subset of X such that A ∩ (X\O) is compact for each open neighborhood O of a non-compact closed sets of X, then A is closed.

In Wilansky proved that the one-point compactification X of X is a KC-space if and only if X is a KC-space and for each S ⊆ X, if S ∩ K is closed, for all closed compact K, then S is closed.

CONCLUSION

Proposition 5.2: Let X be a topological space. Then the following statements are equivalent.

The one-point compactification of X is a KC-space.

X is a KC-space and if A is a subset of X such that A ∩ C is compact, for all compact closed set C of X, then A is compact.

Proof. (i) =⇒ (ii) That X is a KC-space is immediate, since compact subsets of X are compact subset of X. Let A be a subset of X such that A ∩ C is compact, for each compact closed set C of X. Let U be an open cover of K=A ∪ {∞}. Hence there exists U ∈ U such that ∞ ∈ U. Thus X\U is a compact closed set of X, so A ∩ (X\U) is a compact set. Then K is a compact set of X. Now, since Xe is a KC-space, K is a closed set of X, so that A is a closed set of X.

=⇒ (i) Let K be a compact set of X. We discuss two cases.

Case 1: ∞ ∈/ K. Then K is a compact set of X. Thus K is a compact closed set of X. Hence K is a closed set of X.

Case 2: ∞ ∈ K. Let C be compact closed set of X. Then C is a closed set of Xe. Hence K ∩ C is compact set of Xe and of X. Thus K ∩ X is a compact set of X. Since X is a KC-space, K ∩ X is a closed set of X. Then K is a closed set of X. Therefore, X is a KC-space.

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