Oufaska's identity (∀n ε $\mathbb{N}^*$ we have π(2n)+π̄(2n)=n)

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DESCRIPTION

In this article, Oufaska's identity (or Oufaska's equation) asserts that for every natural number n the sum of the prime-counting function π(2n) and the con-counting function π̄(2n) equals n. Oufaska's identity (or Oufaska's equation) has many applications in number theory and its related to one of the famous problems in mathematics for example the twin prime conjecture [1-3].

Notation and reminder

$\mathbb{N}^*$:={1,2,3,4,5,6,7,8,9,10,11,...} The natural numbers.

$\mathbb{N}_{en}$:={0,2,4,6,8,10,12,14,16,18,20,...} The even numbers.

$\mathbb{N}_{con}$:={9,15,21,25,27,33,35,39,45,49,51,...} The composite odd numbers.

$\mathbb{P}$:={2,3,5,7,11,13,17,19,23,29,31,...} The prime numbers.

$\mathbb{P}^*$:={3,5,7,11,13,17,19,23,29,31,37,...} The odd prime numbers.

∀: The universal quantifier and ∃: The existential quantifier.

Card A: The number of elements in A.

A∩B: All elements that are members of both A and B.

A∪B: All elements that are members of both A or B.

∅: The empty set is the unique set having no elements.

Definition 1(*The prime-counting function* $\pi(x)$) . $\forall x > 0$ *we have* $\pi(x) = Card[0, x] \cap \mathbb{P} = Card\{p \le x : p \in \mathbb{P}\}$. *In other words,* $\pi(x)$ *is the number of primes less than or equal* x.

In 1838, Dirichlet observed that $\pi(x)$ can be well approximated by the logarithmic integral function $\mathrm{li}(x) = \int_2^x \frac{dt}{\log t}$ or $\pi(x) \sim \mathrm{li}(x)$ $(x \to \infty)$.
The celebrated prime number theorem, proved independently by de la Vallée Poussin and Hadamard in 1896, states that $\pi(x) \sim \frac{x}{\log x}$ $(x \to \infty)$.

Definition 2(*The prime-counting function* $\pi(2n)$) . $\forall n \in \mathbb{N}^*$ *we have* $\pi(2n) = Card[1, 2n] \cap \mathbb{P} = Card\{p \le 2n : p \in \mathbb{P}\}$. *In other words,* $\pi(2n)$ *is the number of primes less than or equal* $2n$.

Definition 3(*The con-counting function* $\bar{\pi}(2n)$) . $\forall n \in \mathbb{N}^*$ *we have* $\bar{\pi}(2n) = Card[1, 2n] \cap \mathbb{N}_{con} = Card\{p \le 2n : p \in \mathbb{N}_{con}\}$. *In other words,* $\bar{\pi}(2n)$ *is the number of composite odd numbers less than* $2n$.

Definition 4(*The en-counting function* $\bar{\bar{\pi}}(2n)$) . $\forall n \in \mathbb{N}^*$ *we have* $\bar{\bar{\pi}}(2n) = Card[1, 2n] \cap \mathbb{N}_{en} = Card\{p \le 2n : p \in \mathbb{N}_{en}\}$. *In other words,* $\bar{\bar{\pi}}(2n)$ *is the number of even numbers less than or equal* $2n$.

Examples:

For n=1 we have $\pi(2) = 1$ and $\bar{\pi}(2) = 0$ and $\bar{\bar{\pi}}(2) = 1$

For n=2 we have $\pi(4) = 2$ and $\bar{\pi}(4) = 0$ and $\bar{\bar{\pi}}(4) = 2$

For n=3 we have $\pi(6) = 3$ and $\bar{\pi}(6) = 0$ and $\bar{\bar{\pi}}(6) = 3$

For n=4 we have $\pi(8) = 4$ and $\bar{\pi}(8) = 0$ and $\bar{\bar{\pi}}(8) = 4$

For n=5 we have $\pi(10) = 4$ and $\bar{\pi}(10) = 1$ and $\bar{\bar{\pi}}(10) = 5$

For n=6 we have $\pi(12) = 5$ and $\bar{\pi}(12) = 1$ and $\bar{\bar{\pi}}(12) = 6$

For n=7 we have $\pi(14) = 6$ and $\bar{\pi}(14) = 1$ and $\bar{\bar{\pi}}(14) = 7$

For n=8 we have $\pi(16) = 6$ and $\bar{\pi}(16) = 2$ and $\bar{\bar{\pi}}(16) = 8$

Lemma. $\forall n \in \mathbb{N}^*$ we have $\bar{\bar{\pi}}(2n) = n$.

Proof. (Trivial).

Theorem . $\forall n \in \mathbb{N}^*$ *we have* $\pi(2n) + \bar{\pi}(2n) + \bar{\bar{\pi}}(2n) = 2n$.

Proof. Indeed, $\forall n \in \mathbb{N}^*$ we have

$[1, 2n] \cap \mathbb{N}^* := \{1\} \cup \{[1, 2n] \cap \mathbb{N}_{en}\} \cup \{[1, 2n] \cap \mathbb{P}^*\} \cup \{[1, 2n] \cap \mathbb{N}_{con}\}$

where $\{1\} \cap \{[1, 2n] \cap \mathbb{N}_{en}\} \cap \{[1, 2n] \cap \mathbb{P}^*\} \cap \{[1, 2n] \cap \mathbb{N}_{con}\} = \emptyset$

then, $\mathrm{Card}[1, 2n] \cap \mathbb{N}^* = \mathrm{Card}\{1\} + \mathrm{Card}[1, 2n] \cap \mathbb{N}_{en} + \mathrm{Card}[1, 2n] \cap \mathbb{P}^*$

$$+ \mathrm{Card}[1, 2n] \cap \mathbb{N}_{con} = 2n$$

then, $1 + \bar{\bar{\pi}}(2n) + \pi(2n) - 1 + \bar{\pi}(2n) = 2n$

finally, $\pi(2n) + \bar{\pi}(2n) + \bar{\bar{\pi}}(2n) = 2n$.

Corollary (Oufaska's identity). $\forall n \in \mathbb{N}^*$ *we have* $\pi(2n) + \bar{\pi}(2n) = n$.

Proof. $\forall n \in \mathbb{N}^*$ we have $\pi(2n) + \bar{\pi}(2n) + \bar{\bar{\pi}}(2n) = 2n$ and $\bar{\bar{\pi}}(2n) = n$

then, $\pi(2n) + \bar{\pi}(2n) + n = 2n$

finally, $\pi(2n) + \bar{\pi}(2n) = n$.

Remark. $\begin{cases} \bar{\pi}(2n) = 0 \text{ when } n \le 4 \\ \bar{\pi}(2n) \ge 1 \text{ when } n > 4 \end{cases}$.

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Examples:

For $n=1$ we have $\pi(2)+\bar{\pi}(2)=1+0=1$

For $n=2$ we have $\pi(4)+\bar{\pi}(4)=2+0=2$

For $n=3$ we have $\pi(6)+\bar{\pi}(6)=3+0=3$

For $n=4$ we have $\pi(8)+\bar{\pi}(8)=4+0=4$

For $n=5$ we have $\pi(10)+\bar{\pi}(10)=4+1=5$

For $n=6$ we have $\pi(12)+\bar{\pi}(12)=5+1=6$

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