Primorials in Pi

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ABSTRACT

Since at least 1734 (when Euler solved the Basel problem), it's been known for the positive even integers s, the Euler Zeta Function (EZF) $\varsigma(s)$ can be written in terms of the even powers of π^{2k}. I manipulate its form and find lurking (hidden) in it an exquisite and elegant formula for π. Thus, not only does the EZF have π embedded in it, π has embedded in its construction primorials of primes.

Key words: *Primorials; Euler zeta; Basel problem*

INTRODUCTION

For For most people π, i.e. 3.14159…, is the most well-known math constant they can recite to at least a few digits. There are many algorithms that can generate its digits, with varying speed. Using Prime Generator Theory (PGT) we can derive an exquisite formula to compute it, that's been hiding in plain sight (for centuries) that heretofore hadn't been noticed, missed by even the great Leonhard Euler, who probably had the first chance (best mindset) to notice it, but didn't. And its starts with his Zeta function [1-4].

Zeta function $\varsigma(s)$

In contemporary math the Euler/Riemann Zeta function expression is usually written in this form:

$$\varsigma(s)=\sum_{n=1}^{\infty} n^{-s} = \prod_p \frac{1}{1-p^{-s}} \tag{1}$$

But Euler wrote it like this:

$$\varsigma(s)=\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \frac{p^s}{p^{-s}-1} \tag{2}$$

Written in primorial form it's:

$$\varsigma(s) = \prod_p \frac{p^s}{p^s-1} = \frac{p_n^s\#}{(p_n^s-1)\#} \tag{3}$$

For s = 2 we get:

$$\varsigma(2) = \frac{p_n^2\#}{(p_n^2-1)\#} \tag{4}$$

But $\varsigma(2)=\pi^2$ / 6, and p_n^2 # is $(p_n\#)^2$, which now gives us this exquisite formula for π.

$$\frac{\pi^2}{6} = \frac{(p_n\#)^2}{(p_n^2-1)\#} \tag{5}$$

$$\pi = \frac{\sqrt{6}\, p_n\#}{(\sqrt{p_n^2-1})\#} = (3\,\#)^{1/2}\, \frac{p_n\#}{(p_n^2-1)^{1/2}\#} \tag{6}$$

And now we see a simple formula for π hidden in the background of the Zeta function! We see we can represent (and calculate) π strictly with primorials, i.e. consecutive prime factors. We'll further see not only does lurk within the $\varsigma(2k)$ values, but the primorials also π lurk within the construction of π.

But we don't have to stop with $\varsigma(2)$, as each expression for $\varsigma(2k)$ has a factor of π^{2k} in it.

For s = 2k:

$$\varsigma(2k) = (-1)^{k+1} \frac{B_{2k}\, 2^{2k}}{2(2k)!} \pi^{2k} \tag{7}$$

The B_{2k} are the 2k-th Bernoulli numbers. Here are the first 8 expressions for $\varsigma(2k)$ [5-11].

$$\varsigma(2)=\frac{\pi^2}{6} \qquad \varsigma(4)=\frac{\pi^4}{90} \qquad \varsigma(6)=\frac{\pi^6}{945} \qquad \varsigma(8)=\frac{\pi^8}{9450}$$

$$\varsigma(10)=\frac{\pi^{10}}{93555} \qquad \varsigma(12)=\frac{691\pi^{12}}{638512875} \qquad \varsigma(14)=\frac{2\pi^{14}}{18243225} \qquad \varsigma(16)=\frac{3617\pi^{16}}{325641566250}$$

I'll show we can compute π to increasing accuracy with primorials, using its generalized form:

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$$\pi = C_{z2k}^{1/2k} \prod_p \frac{p_n}{(p_n^{2k}-1)^{1/2k}} = C_{z2k}^{1/2k} \frac{p_n\#}{(p_n^{2k}-1)^{1/2k}\#} \tag{8}$$

where the C_{z2k} are the constant rational inverse coefficients of π^{2k} from the $\varsigma(2k)$ expressions.

$$C_{z2k} = \frac{6}{1} = 6 \qquad C_{z4} = \frac{90}{1} = 1 \qquad C_{z6} = \frac{945}{1} = 945 \qquad C_{z8} = \frac{9450}{1} = 9450$$

$$C_{z10} = \frac{93555}{1} = 93555 \qquad C_{z12} = \frac{638512875}{691} \qquad C_{z14} = \frac{18243225}{2} \qquad C_{z16} = \frac{325641566250}{3617}$$

With the C_{z2k} having form: $C_{z2k} = (-1)^{K+1}\dfrac{(2k)!}{2^{2k-1}B_{2k}}$ (9)

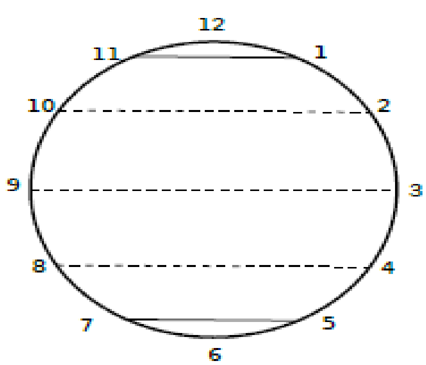
What we will discover is that the C_{z2k} coefficients have embedded within them the value of π, to increasing digits of accuracy. From their starting approximations for π, the primorials boost the number of accurate digits higher, as more primes are used in their construction. We'll also discover that from the factorization of the C_{z2k} numerators we can reconstruct their written forms as factors of primorials.

Geometric interpretation using PGT

Let's see how to geometrically understand this conceptually, from the perspective of PGT.

As explained in [1], [2], [3] Prime Generators break the number line into modular groups of size $p_n\#$ integers, which contain $(p_n - 1)\#$ integer residues, along which all the primes not a factor of $p_n\#$ exist. As we increase the modular group size by p_n we increase the number of residues by $(p_n - 1)$. . This has the effect of squeezing the primes into a smaller and smaller percentage of the integer number space. It's essentially the same process Euler used to squeeze out all the composites in the reciprocal integer form (1), (2) of the Zeta function to create his multiplicative prime (primorial) form (3).

Useful for our purposes here, we can model the periodicity of the modular groups with a clock.



Using our generator clock model we can conceptualize the geometric meaning of the expression for π.

$$\pi = C_{z2k}^{1/2k} \frac{p_n\#}{(p_n^{2k}-1)^{1/2k}\#} \tag{10}$$

From geometry: $\pi = \dfrac{c}{d} = \dfrac{c}{2r}$ (11)

where $r = c/2\pi = c/\tau$, with $(tau)\,\tau = 2\pi$. Thus when we take generators of length $p_n\#$ integers, and fold them into, and model them as clocks (modular circles), $c = p_n\#$ is the circumference of these circles, which increase by factors of p_n for each larger generator. Thus we get these geometric relationships:

$$c = p_n\# \qquad d = \frac{(p_n^{2k}-1)^{1/2k}\#}{C_{z2k}^{1/2k}} \qquad r = \frac{(p_n^{2k}-1)^{1/2k}\#}{2C_{z2k}^{1/2k}} \tag{12}$$

$$c^{2k} = p_n^{2k}\# \qquad d^{2k} = \frac{(p_n^{2k}-1)\#}{C_{z2k}} \qquad r^{2k} = \frac{(p_n^{2k}-1)\#}{2^{2k}C_{z2k}} \tag{13}$$

Thus we see the modular diameters and radii expressions are the (principal) 2k-th roots of primordial expressions. Thinking about this more extensively, this suggests there may be complex roots, which we know come as complex conjugate pairs. This would be consistent with the fact that the generator residues come as modular complement pairs. We'll also see for the $p_n, d_n \Box p_n / \pi_{z2k}$ *and* $r_n \Box p_n / \tau_{z2k}$.

I've only scratched the surface here, but I'll suspend going further down this rabbit hole of analysis, as it's diverging from the principal purpose of this paper. However, it presents itself as an interesting area Of math to explore and develop, and I encourage others to vigorously pursue it if desired.

Numerical analysis

Compared to other methods for generating π, the presented method is much simpler to understand and remember. And from a Number Theory point of view, it also has a conceptual and numerically pleasing elegance, which I will show and explain. To demonstrate its utility and performance I provide software code to generate some results of its accuracy and convergence speed for the first few C_{z2k} coefficients.

From this form of the formula:

$$\pi = C_{z2k}^{1/2k} \prod_p \frac{p_n}{(p_n^{2k}-1)^{1/2k}} \tag{14}$$

We expand it into:

$$\pi = C_{z2k}^{1/2k} \cdot \frac{2}{(2^{2k}-1)^{1/2k}} \cdot \frac{3}{(3^{2k}-1)^{1/2k}} \cdot \frac{5}{(5^{2k}-1)^{1/2k}} \cdots \tag{15}$$

In fact, this is the form of the algorithm the software code uses to numerically compute it.

Notice in the factors $(p_n^{2k} - 1)^{1/2k}$ we're raising each to a power 2k, then bringing one less than that p_n value back down to be almost (but less than) p_n. Using $p_2 = 3$ as an example, we can see the process.

$$(3^2 - 1)^{1/2} = (9-1)^{1/2} = 8^{1/2} = 2.82842\ldots$$

$$(3^4-1)^{1/4}=(81-1)^{1/4}=80^{1/4}=2.990697...$$

$$(3^6-1)^{1/6}=(729-1)^{1/6}=728^{1/6}=2.99931...$$

$$(3^8-1)^{1/8}=(6561-1)^{1/8}=728^{1/6}=6560^{1/8}=2.99994...$$

As 2k increases $(p_n^{2k}-1)^{1/2k}$ becomes increasingly closer to p_n. If we set p_{n-} to be $(p_n^{2k}-1)^{1/2k}$ then the primorial ratios p_n / p_{n-} are always > 1 but can be made arbitrarily close to 1, as $2k\to\infty$.

Thus as $2k\to\infty$:

$$\prod_p \frac{p_n}{p_{n-}}=\frac{2}{1.999...}\cdot\frac{3}{2.999...}\cdot\frac{5}{4.999...}\cdot\frac{7}{6.999...}\cdots\to 1.0000... \qquad (15)$$

So if the primorial ratios are marching in unison toward 1 where do we get π from? Well, there's only one place left its digits can come from. And this is what we discover, apparently missed by even Euler.

$$C_{z2}^{1/2}=6^{1/2}=2.449489...$$

$$C_{z4}^{1/4}=90^{1/4}=3.080070...$$

$$C_{z6}^{1/6}=945^{1/6}=3.132602...$$

$$C_{z8}^{1/8}=9450^{1/8}=3.139995...$$

$$C_{z10}^{1/10}=93555^{1/10}=3.141280...$$

$$C_{z12}^{1/12}=(638512875\ /\ 691)^{1/12}=3.141528...$$

We can theoretically get arbitrary convergence with a few (or just p_1 = 2) primes. However in the real world, at least with using personal computers, calculators, etc, we will soon hit the wall in reaching the limit on the number of digits floating point implementations can accurately represent. But that is an implementation issue true for all numerical (floating point) operations performing computations with small numbers. However, software algebra systems like Pari/GP [10], et al, are specifically designed to provide arbitrary precision in such situations, which I'll use to show some calculations.

Values for $\mathbf{C_{z2k}}$

We've previously seen that:

$$\varsigma(2k)=(-1)^{k+1}\frac{B_{2k}2^{2k}}{2(2k)!}\pi^{2k} \qquad (17)$$

and therefore the $\mathbf{C_{z2k}}$ are:

$$\mathbf{C_{z2k}}=(-1)^{k+1}\frac{(2k)!}{2^{k-1}B_{2k}} \qquad (18)$$

From [5], $A_n\varsigma(2k)=B_n\pi^{2k}$, and thus $C_{z2k}=\frac{A_n}{B_n}$, where A_n and B_n are positive integers for n even.

There are lists of some of them already pre-computed, or we can compute them, using online resources.

Sequence lists for the first 250 can be found on the On-Line Encyclopedia of Integer Sequences (OEIS) website, with the A_{2k} – A002432 sequence and B_{2k} – A046988 sequence at [11].

We can get many, many more using the WolframAlpha math engine.

As an example, putting in the searchbar zeta(18), returns (43867 π^18) / 38979295480125, making C_{z18} = 38979295480125 / 43867. These numbers grow fast. For zeta(250) we get for A_{250} and B_{250},

A_{250}=7577834251451999039514401422585053122879168525469783889771482580471299697556140678102433420069852311782394668455019655075279786731641291340665903313976613292986707694694276386213013321688626077273626360722083755519960623995664925415614342891786954820102266673859922242112137297234692568084815586417724540368564961186064006209286287958532390164917820776212155836535163133842859024841970283503655991808045655488967895507812

5

B_{250}=3909101330895614339970586848854445032806776798697690058731856271636606737446563047969362779198681815937490997579747372978638362077570964830350055369483897650216572621487022225127006100471782640902357514653698268264535930011285025251204753835385516031169725748375567261264716061757515293916631176

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and then $C_{z250}^{1/250}=(A_{250}\ /\ B_{250})^{1/250}$ (which I'll show later gives the first 78 accurate digits of π) and from there we can boost the number of digits further by the EZF primorials multiplications shown in (15), which we see from the short Ruby code that follows, starting with the first few C_{z2k} values.

From just looking at these values you can begin to image the scale of their sizes for larger coefficients. Also as their values increase, they will contain more and more accurate digits of π. And as there are an unending number of C_{z2k} coefficients, there are an unending number of π digits they will represent, which can then be boosted to even higher accuracy by multiplying them by the EZF primorial ratios.

Thus you can see (even feel) this deep structural connection between π and the primes, and primes to circles, and in general to the concept of periodicity of functions, derived from the *Euler Zeta Function* (Table 1).

Below is Ruby code to generate π to 15 digits (when capable) using the coefficients for $C_{z2}-C_{z16}$.

```
require "primes/utils"                        # Load primes-utils RubyGem

def pi_Z2k(k2, cz2k, primes)
 pi, exp = 1.0, 1.0/k2
 primes.each do |p|
    pi *= p / (p**k2 - 1)**exp
 end
  pi * cz2k**exp end

# Example inputs for Zeta(8) nth = 18          # Select number of primes to use
nth_prime = nth.nthprime                       # Set prime value of nth prime
n_primes = nth_prime.primes  k2, cz2k = 8,     # Generate array of first n primes
9450                                           # Set Zeta(8) parameters

puts "\nUsing #{nth} primes up to               #{nth_prime}
pi = pi_Z2k(k2, cz2k, n_primes)                # Using 18 primes uo to 61
puts "pi_Z#{k2} = #{pi} \n"                    # pi_Z8 = 3.141592653589792
```

TABLE 1
This table shows the speed of convergence up to pi_Z16. On my laptop using Ruby, I was able to get up to 15 significant digits of accuracy until the fractions got too small to generate more accurate digits.

Pi digits	pi_Z2 m primes	pi_Z4 m primes	pi_Z6 m primes	pi_Z8 m primes	pi_Z10 m primes	pi_Z12 m primes	pi_Z14 m primes	pi_Z16 m primes
3.	2							
3.1	5	1						
3.14	38		1					
3.141	76	3						
3.1415	301	5	2	1	1			
3.14159	516	10		2				
3.141592	16,663	14	4	3		1		
3.1415926	142,215	26	6		2		1	1
3.14159265	1,534,367	51	9	4	3	2		
3.141592653		80	11	5				
3.1415926535		132	15	6	4	3	2	
3.14159265358		240	21	8	5			2
3.141592653589		481	30	10	6	4	3	
3.1415926535897		837	40	13	7	5		3
3.14159265358979				18		6	4	4

Using arbitrary precision software we'd see we can boost the initial true digits to arbitrary size by using more primes. Thus we can get arbitrary digits from the   alone, and from C_{z2k} using the *EZF* primorials. This approach for generating π may be interesting to compare to the Chudnovsky algorithm [6], which (as of March 21, 2022) computed it to a record 100 trillion digits, and in general, to test the speed and numerical accuracy of super computers, *et al*.

Factoring into primorials

The C_{z2k} numerators A_{2k} can be written as primorial factors, first factoring them and then completing their primorials from the prime factors, and including factors of 2 in the denominator when necessary.

C_{z2} is easy: $C_{z2} = 6 = 2\cdot3 = 3\#$.

For C_{z4}: $C_{z4} = 90 = 2\cdot3^2\cdot5 = 3\cdot(2\cdot3\cdot5) = 3\cdot5\# = \frac{2\cdot3}{2}\cdot5\# = \frac{3\#5\#}{2\#}$

The process continues in this straightforward manner, and can be done visually by just completing the primorial for the largest remaining prime factor, always accounting for factors of 2 in the denominator.

$$C_{z6} = 945 = 3^3\cdot5\cdot7 = 3^2\cdot\frac{2\cdot3\cdot5\cdot7}{2} = \frac{3^2\cdot7\#}{2} = \frac{2^2\cdot3^2}{2^2}\cdot\frac{7\#}{2} = \frac{(2\cdot3)^2\cdot7\#}{2^3} = \frac{(3\#)^2\,7\#}{(2\#)^3}$$

With practice, you can just write down the primorials after each prime factor step, as shown here.

$$C_{z20} = \frac{1531329465290625}{174611} = \frac{(3\#)^4\,(5\#)^3\,11\#19\#}{(2\#)^9\,174611}$$

$$1531329465290625 = 3^9 \cdot 5^5 \cdot 7^2 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19$$

$$= \frac{3^8 \cdot 5^4 \cdot 7 \cdot 11 \cdot 19\#}{2}$$

$$= \frac{3^7 \cdot 5^3 \cdot 11\# \cdot 19\#}{2^2}$$

$$= \frac{3^4 \cdot (5\#)^3 \cdot 11\# \cdot 19\#}{2^9}$$

$$= \frac{(3\#)^4 \cdot (5\#)^3 \, 11\# \, 19\#}{(2\#)^9}$$

While there can be different representations for C_{z2k} the primorial factorizations reveal their inherent structure based upon the building up of small primes.

Thus, while $\varsigma(26)$ can be written as:

$$\varsigma(26) = \frac{2^{24} \cdot 76977927 \cdot \pi^{26}}{27!} \rightarrow C_{z26} = \frac{27!}{2^{24} \cdot 76977927}$$

it doesn't reveal its primes structure written as:

$$C_{z26} = \frac{(3\#)^5 (5\#)^3 \; 7\#11\#23\#}{(2\#)^{11} \, 1315862}$$

Another amazing property you'll notice of the primorial forms of the A_{2k} integers is that the highest primorial prime value p_m of their factoring is the closest prime less than or greater than the value *2k*.

Let's put all the pieces together and show the computation of $C_{z250}^{1/250}$ to 100 digits, giving 78 digits of π .

$$\begin{aligned} A_{250} = {} & 7577834251451999039514401422585053122879168525469783889771482580471296975556 \\ & 1406781024334200698523117823946684550196550752797867316412913406659033139 76 \\ & 6132929867076946942763862130133216886260772736263607220837555199606239956 64 \\ & 9254156143428917869548201022666738599222421121372972346925680848155864177 24 \\ & 5403685649611860640062092862879585323901649178207762121558365351631338428 59 \\ & 02484197028350365599180804565548896789550781 25 \end{aligned}$$

$$\begin{aligned} A_{250} = {} & 3^{124} \cdot 5^{59} \cdot 7^{40} \cdot 11^{25} \cdot 13^{20} \cdot 17^{14} \cdot 19^{13} \cdot 23^{10} \cdot 29^8 \cdot 31^8 \cdot 37^6 \cdot 41^6 \cdot 43^5 \cdot 47^5 \cdot 53^4 \cdot 59^4 \cdot \\ & 61^4 \cdot 67^3 \cdot 71^3 \cdot 73^3 \cdot 79^3 \cdot 83^3 \cdot 89^2 \cdot 97^2 \cdot 101^2 \cdot 103^2 \cdot 107^2 \cdot 109^2 \cdot 113^2 \cdot 127 \cdot 131 \cdot 137 \cdot \\ & 139 \cdot 149 \cdot 151 \cdot 157 \cdot 163 \cdot 167 \cdot 173 \cdot 179 \cdot 181 \cdot 191 \cdot 193 \cdot 197 \cdot 199 \cdot 211 \cdot 223 \cdot 227 \cdot 229 \cdot \\ & 233 \cdot 239 \cdot 241 \cdot 251 \end{aligned}$$

$$A_{250} = \frac{(3\#)^{65}(5\#)^{19}(7\#)^{15}(11\#)^5(13\#)^6 17\#(19\#)^3(23\#)^2(31\#)^2 41\#47\#61\#83\#113\#251\#}{(2\#)^{124}}$$

$$\begin{aligned} B_{250} = {} & 3909101330895614339970586848854445032806776798697690058731856271636606737 44 \\ & 6563047969362779198681815937490997579747372978638362077570964830350055369 48 \\ & 3897650216572621487022225127006100471782640902357514653698268264535930011 28 \\ & 502525120475383538551603116972574837556726126471606175751529391663117616 \end{aligned}$$

```
Pari/GP calculator output (edited)
? a250 =
7577834251451999039514401422585053122879168525469783889771482580471296975561406781
0243342006985231178239466845501965507527978673164129134066590331397661329298670769
4694276386213013321688626077273626360722083755519960623995664925415614342891786954
8201022666738599222421121372972346925680848155864177245403685649611860640062092862
8795853239016491782077621215583653516313384285902484197028350365599180804565548896
78955078125

? b250 =
3909101330895614339970586848854445032806776798697690058731856271636606737446563047
9693627791986818159374909975797473729786383620775709648303500553694838976502165726
2148702222512700610047178264090235751465369826826453593001128502525120475383538551
603116972574837556726126471606175751529391663117616

? \p 100
realprecision = 115 significant digits (100 digits displayed)

? cz250 = 1.0*a250/b250
1.938510570590873164781494893481210599307810912050800731842107930373389215123813263
067816173293718294 E124

? cz250^(1/250)
3.141592653589793238462643383279502884197169399375105820974944592307816406286202053
009170736283102349
```

Growth of Pi digits for C_{z2k}

Two natural questions are: 1) for a given C_{z2k} how many accurate π digits will it contain?, and 2) what C_{z2k} will first give a certain number of digits? The plot below shows the π digits for the first 250 C_{z2k} .

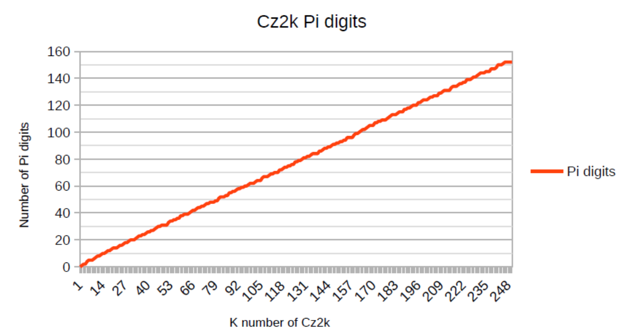

Figure 1) *We see there's a clear linear relationship, thus we can create the equation of its line.*

We see there's a clear linear relationship, thus we can create the equation of its line, $y = mk + C$.

From the data, at *k=1*, digits =0, and *k=250*, digits =152, from which we can get the slope *m*.

Therefore the slope is: $\quad \mathrm{m} = 152 \;/\; 250 \;=\; 0.608 \qquad (19)$

and the line equation: $\quad y \;=\; 0.608k \qquad (20)$

We now have a deterministic way to answer these two questions about the growth of π digits in C_{z2k} .

Thus, for the 1000th coefficient, from $y \;=\; 0.608(1000), C_{z2000}$ gives about the first 608 digits of π , and from $k = 1000 \;/\; 0.608$, we see that to get the first 1000 digits of π we need to use up to about C_{z3290} .

Thus, though the integers A_{2k} *and* B_{2k} grow exponentially, their ratios 2k-th roots grow linearly to π .

FURTHER RESEARCH

We also know for some number z there are n root values for $z^{1/n}$, some as complex conjugate pairs.

Thus for example, $C_{z8}^{1/8} = 9450^{1/8}$ gives us 7 more roots besides the principal root π approximation.

(3.14036879+0.0i), (2.22057607+2.22057607i), (0.0+3.14036879i), (-2.22057607+2.22057607i)

(-3.14036879+0.0i), (-2.22057607-2.22057607i), (0.0-3.14036879i), (2.22057607-2.22057607i)

What do the other roots mean in this context (if any), especially the complex ones? How do they fit in?
These, and other questions, may open up new areas of research pursuits, and more amazing discoveries.

CONCLUSION

Using Prime Generator Theory as the mathematical|conceptual framework to start from, I looked at Euler's Zeta function differently since when he solved the Basel problem in 1734. Discovered lurking within its structure, is a simple/elegant formula to compute π to arbitrary accuracy, previously missed.

Specifically for s = 2k, we see π is embedded in the coefficients 2k-th roots to arbitrary accuracy, which can then be boosted to higher arbitrary accuracy by primorial multiplications. Their numerators can be factored into consecutive small primes, and written as primorial factors, whose largest is the closest prime less/greater than 2k. Finally, we find we can predict the number of digits for each coefficient, and which coefficients will provide a desired number of digits. Thus we find that primorials (primes) are inextricably linked to π, and thus to the geometry of circles, which heretofore was totally unexpected.

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9. Zakiya J. Primes-utils handbook. 2016.
10. Pari/GP software algebra system.
11. $\varsigma(2k)\ A_{2k} \mid B_{2k}$ integer lists – $[A_{2k}]$ and $[B_{2k}]$.

List of Primorials in Pi from first 25 C_{z2k} constants

$$C_{z2} = 6 = 3\#$$

$$C_{z4} = 90 = \frac{3\#5\#}{2\#}$$

$$C_{z6} = 945 = \frac{(3\#)^2\ 7\#}{(2\#)^3}$$

$$C_{z8} = 9450 = \frac{3\#5\#7\#}{(2\#)^2}$$

$$C_{z10} = 93555 = \frac{(3\#)^4\ 11\#}{(2\#)^5}$$

$$C_{z12} = \frac{638512875}{691} = \frac{(3\#)^3\ 5\#7\#13\#}{(2\#)^6\ 691}$$

$$C_{z14} = \frac{18243225}{2} = \frac{(3\#)^4\ 5\#13\#}{(2\#)^7}$$

$$C_{z16} = \frac{325641566250}{3617} = \frac{(3\#)^3(5\#)^2\ 7\#17\#}{(2\#)^6\ 3617}$$

$$C_{z18} = \frac{38979295480125}{43867} = \frac{(3\#)^6(7\#)^2\ 19\#}{(2\#)^9\ 43867}$$

$$C_{z20} = \frac{1531329465290625}{174611} = \frac{(3\#)^4(5\#)^3\ 11\#19\#}{(2\#)^9\ 174611}$$

$$C_{z22} = \frac{13447856940643125}{155366} = \frac{(3\#)^6\ 5\#(7\#)^2\ 23\#}{(2\#)^{10}\ 155366}$$

$$C_{z24} = \frac{201919571963756521875}{236364091} = \frac{(3\#)^6\ 5\#(7\#)^2\ 13\#23\#}{(2\#)^{11}\ 236364091}$$

$$C_{z26} = \frac{11094481976030578125}{1315862} = \frac{(3\#)^5(5\#)^3\ 7\#11\#23\#}{(2\#)^{11}\ 1315862}$$

$$C_{z28} = \frac{564653660170076273671875}{6785560294} = \frac{(3\#)^7(5\#)^4\ 7\#13\#29\#}{(2\#)^{14}\ 6785560294}$$

$$C_{z30} = \frac{56608788046690826740700156 25}{6892673020804}$$

$$= (3\#)^9\ 5\#(7\#)^2\ 11\#13\#31\#/(2\#)^{15}\ 6892673020804$$

$$C_{z32} = \frac{62490220571022341207266406250}{7709321041217}$$

$$= (3\#)^7(5\#)^4(7\#)^2\ 17\#31\#/(2\#)^{14}\ 7709321041217$$

$$C_{z34} = \frac{121304545814337485872928906 25}{151628697551}$$

$$= (3\#)^9(5\#)^3\ 7\#11\#13\#31\#/(2\#)^{16}\ 151628697551$$

$$C_{z36} = \frac{20777977561866588586487628662044921875}{26315271553053477373}$$

$$= (3\#)^9(5\#)^3(7\#)^3\ 13\#19\#37\#/(2\#)^{18}\ 26315271553053477373$$

$$C_{z38} = \frac{2403467618492375776343276883984375}{308420411983322}$$

$$= (3\#)^{10}(5\#)^3(7\#)^2\ 11\#17\#37\#/(2\#)^{18}\ 308420411983322$$

$$C_{z40} = \frac{200804311722896388267984011283905566406 25}{261082718496449122051}$$

$$= (3\#)^9(5\#)^5\ 7\#11\#13\#19\#41\#/(2\#)^{19}\ 261082718496449122051$$

$$C_{z42} = \frac{23077891898189601277125944278646674277343 75}{3040195287836141605382}$$

$$= (3\#)^{11}(5\#)^2(7\#)^4\ 13\#19\#43\#/(2\#)^{20}\ 3040195287836141605382$$

$$C_{z44} = \frac{379136795470257735267069084577766791699218 75}{5060594468963822588186}$$

$$= (3\#)^{10}(5\#)^4(7\#)^3\ 13\#23\#43\#/(2\#)^{20}\ 5060594468963822588186$$

$$C_{z46} = \frac{7670102214448301053033358480610212529462890625}{10373062810328907187442 8}$$

$$= (3\#)^{12}(5\#)^4(7\#)^2\ 11\#13\#19\#47\#/(2\#)^{22}\ 103730628103289071874428$$

$$C_{z48} = \frac{40936486033842749965196989214788795801622866699218 75}{56094033689978176862491275 47}$$

$$= (3\#)^{12}(5\#)^4(7\#)^3\ 13\#17\#23\#47\#/(2\#)^{23}\ 5609403368997817686249127547$$

$$C_{z50} = \frac{28525877145754676446336363525237441418325436523437 5}{39604576419286371856998202}$$

$$= (3\#)^{13}(5\#)^2(7\#)^3(11\#)^2\ 13\#23\#47\#/(2\#)^{23}\ 39604576419286371856998202$$

$$C_{z2}^{1/2} = 2.449489742783178$$

$$C_{z4}^{1/4} = 3.080070288241023$$

$$C_{z6}^{1/6} = 3.132602581012435$$

$$C_{z8}^{1/8} = 3.139995141295907 3$$

$$C_{z10}^{1/10} = 3.1412803693973714$$

$$C_{z12}^{1/12} = 3.1415282368670168$$

$$C_{z14}^{1/14} = 3.1415789099913694$$

$$C_{z16}^{1/16} = 3.1415896529495364$$

$$C_{z18}^{1/18} = 3.1415919871238964$$

$$C_{z20}^{1/20} = 3.1415925037418626$$

$$C_{z22}^{1/22} = 3.1415926195391455$$

$$C_{z24}^{1/24} = 3.1415926457870995$$

$$C_{z26}^{1/26} = 3.141592651789231$$

$$C_{z28}^{1/28} = 3.1415926531718115$$

$$C_{z30}^{1/30} = 3.141592653492265$$

$$C_{z32}^{1/32} = 3.141592653566935$$

$$C_{z34}^{1/34} = 3.1415926535844148$$

$$C_{z36}^{1/36} = 3.141592653588523$$

$$C_{z38}^{1/38} = 3.1415926535894925$$

$$C_{z40}^{1/40} = 3.141592653589722$$

$$C_{z42}^{1/42} = 3.1415926535897762$$

$$C_{z44}^{1/44} = 3.141592653589789$$

$$C_{z46}^{1/46} = 3.1415926535897922$$

$$C_{z48}^{1/48} = 3.1415926535897927$$

$$C_{z50}^{1/50} = 3.141592653589793$$