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Sum (2,M)-double fuzzifying continuity and characterizations of (2,M)-double fuzzifying topology

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ABSTRACT

(2,M)-double fuzzifying topology is a generalization of (2,M)- fuzzifying topology and classical topology. Motivated by the study of (2,M)- fuzzifying topology introduced by Höhle for fuzzifying topology. The main motivation behind this paper is introduce (2,M)-double fuzzifying topology as tight definition and a generalization of (2,M)- fuzzifying topology. Also, study structural properties of (2,M)-double fuzzifying continuous mapping, (2,M)-double fuzzifying quotient mapping, (2,M)-double fuzzifying operator, (2;M)-double fuzzifying totally continuous mapping and define an (2,M)-double fuzzifying Interior (closure) operator. The respective examples of these notions are investigated and the related properties are discussed. On the other hand, a characterization of (2,M)-fuzzifying topology by (2,M)-fuzzifying neighborhood system, where M is a completely distributive, was given in Höhle (2). We extended this defination and others to (2,M)-double fuzzifying topology. As an application of our results, we get characterizations of a (2,M)-double fuzzifying topology by these new notions. These characterizations do not exist in literature before this work. These concepts will help in verifying the existing characterizations and will be useful in achieving new and generalized results in future works.

Key Words: *(2,M)-double fuzzifying topology; (2,M)-double fuzzifying continuous mapping; Characterizations of (2,M)-double fuzzifying topology*

THe uncertainty appeared in economics, engineering, environmental science, medical science and social sciences and so many other applied sciences is too complicated to be solved by traditional mathematical frameworks. The concept of (2,M)- fuzzifying topology appeared in Höhle (1,2) under the name "(2,M)-fuzzy topology" (cf. Definition 4.6, Proposition 4.11 in (2)) where L is a completely distributive complete lattice. In the case of $L=[0,1]$ this terminology traces back to Ying (3-5) who studied the fuzzifying topology and elementarily developed fuzzy topology from a new direction with semantic method of continuous valued logic. Fuzzifying topology (resp. (2,M)-fuzzifying topology) in the sense of Ying (resp., Höhle) was introduced as a fuzzy subset (resp., an M-Fuzzy subset) of the power set of an ordinary set. (2,M)- fuzzifying topology is a kind of new mathematical model for dealing with uncertainty from a parameterization point of view. Also, Höhle's in (2) from Theorems 1.4.2, 14.3, the concepts of (2,M)-fuzzifying topology and (2,M)- fuzzifying neighborhood system are equivalent notions. In my work we extended the notions of (2,M)- fuzzifying topology into (2,M)-double fuzzifying topology and studied the related properties and gave many valuable results for this theory which can be used as a generic mathematical tool for dealing with uncertainties. In the present paper, we apply the (2,M)-double fuzzifying topology, to M-double fuzzifying continuous mappings, M-double fuzzifying quotient mapping, (2,M)-double fuzzifying totally continuous mapping and define an (2,M)-double fuzzifying Interior (closure) operator. We extend and studied the notions of (2,M)-double fuzzifying neighborhood, M-double fuzzy contiguity relations, and M-double fuzzifying closure (interior) operator. Then our generalization of Höhle (2-5) results is obtained if we prove that M-double fuzzifying contiguity relation, (2,M)-double fuzzifying topology, (2,M)-double fuzzifying neighborhood system and M-double fuzzifying closure (interior) operator relation are equivalent notions. These characterizations do not exist in literature before this work. The basic properties of these notions are studied and characterizations of these concepts are discussed in detail. In Section 1.1, we introduce a survy about the definitions used in the article. In Section 1.2, we study

structural properties of (2,M)-double fuzzifying continuous mapping, (2,M)-double fuzzifying quotient mapping, (2,M)-double fuzzifying operator. In Section 1.3, we discuss (2,M)-double fuzzifying totally continuous and define an (2,M)-double fuzzifying Interior (closure) operator. The respective examples of these notions are investigated and the related properties are discussed. The basic properties of these notions are studied. On the other hand, in Section 1.4, a cheracterization of (2,M)-double fuzzifying topology by (2,M)-double fuzzifying neighborhood system, M-double fuzzifying contiguity relation, M-double fuzzifying interior operator are introduced, where M is a complete residuated lattice. For this paper M is complete residuated lattice and for more details see (6-12).

The following Definitions and Results introduced by Höhle (2).

Definition 1.1

The double negation law in a complete residuated lattice L is given as follows: $L, (a \to \perp) \to \perp = a.$

Definition 1.2

A structure $(L, \vee, \wedge, *, \to, \perp, \top)$ is called a strictly two-sided commutative quantale iff

(1) $(L, \vee, \wedge, \perp, \top)$ is a complete lattice whose greatest and least element are $\perp, \top$ respectively,

(2) $(L, *, \top)$ is a commutative monoid,

(3) (a) $*$ is distributive over arbitrary joins, i.e.,

$$a * \bigvee_{j \in J} b_j = \bigvee_{j \in J} (a * b_j) \forall a \in L, \forall \{b_j \in J\} \subseteq L,$$

(b) $\to$ is a binary operation on L defined by:

$$a \to b = \bigvee_{\lambda * a \leq b} \lambda \forall a, b \in L.$$

Definition 1.3

A structure $(L, \vee, \wedge, *, \to, \perp, \top)$ is called a complete MV- algebra iff the

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following conditions are satisfied:

1. $(L,\vee,\wedge,*,\rightarrow,\perp,\top)$ is a strictly two-sided commutative quantale,
2. $\forall a,b\in L,(a\rightarrow b)\rightarrow b=a\vee b.$

Definition 1.4

Let $x\in X$. The fuzzifying neighbourhood system of x, denoted by $N_x\in I^{P(X)}$, is defined as follows: $N_x(A)=\bigvee_{x\in B\subseteq A}\mathcal{T}(B)$.

Definition 1.5

Let X be a nonempty set. An element $c\in L^{X\times P(X)}$ is called an M-fuzzy contiguity relation on X iff c fulfills the following axioms:

$(c_1)c(x,\phi)=\perp\ \forall x\in X.$

$(c_2)c(x,A\cup B)=c(x,A)\vee c(x,B),$ (Distributivity),

$(c_3)c(x,A)=\top,$ whenever $x\in A,$

$(c_4)(\bigwedge_{y\in B}c(y,A))\wedge c(x,B)\le c(x,A).$ (Transitivity).

Theorem 1.1

Let $(X,\mathcal{T})$ be an (2,M)-fuzzifying topological space, and let L satisfies the completely distributive law then the (2,M)-fuzzifying neighborhood system $(\varphi_x)x\in X$ satisfies the following conditions:

$(f_1)\varphi_x(X)=\top,\forall x\in X,$ (Boundary conditions)

$(f_2)\varphi_x(A\cap B)=\varphi_x(A)\wedge\varphi_x(B),$ (Intersection property)

$(u_3)\varphi_x(A)=\perp$ whenever $x\notin A$

$(u_4)\varphi_x(A)\le\bigvee_{y\notin B}(\varphi_y(A)\wedge\varphi_x(B))\forall B\in P(X).$ Furthermore $\mathcal{T}(A)=\bigwedge_{x\in A}\varphi_x(A)\forall A\in P(X).$

Theorem 1.2

Let L satisfies the completely distributive law and Let $(\varphi_x)x\in X$ be a system satisfies the properties $(f_1),(f_2),(u_3),(u_4)$ in Theorem 1.4.2 above. Then $(\varphi_x)x\in X$ induces an (2,M)-fuzzifying topology $\mathcal{T}$ on X by $\mathcal{T}(A)=\bigwedge_{x\in A}\varphi_x(A)\forall A\in P(X).$ Moreover the following formula holds $\varphi_x(A)=\bigvee_{x\in B\subseteq A}\mathcal{T}(B).$

Theorem 1.3

Let $(L,\le,*)$ be a complete MV-algebra and $\odot=\wedge.$ further more let $(L,\le)$ be a completely distributive lattice complete MV -algebra. Then (2,M)-fuzzifying topologies, M-fuzzy contiguity relations and stratified and transitive M-topologies are equivalent concepts.

Definition 1.6

Let X be a nonempty set. A map $()^\circ:2^X\rightarrow L^X$ is called an M-fuzzifying interior operator if $()^\circ$ satisfies the following conditions:

$(1^0)(X)^\circ=1_X,$

$(2^0)(A\cap B)^\circ=(A)^\circ\wedge(B)^\circ,$

$(3^0)(A)^\circ\le A,$

$(4^0)(A)^\circ(x)\le\bigvee_{y\notin B}((A)^\circ(y)\vee(B)^\circ(x)).$

Definition 1.7

[1] Let X be a nonempty set and let $P(X)$ be the family of all ordinary subsets of X. An element $\mathcal{T}\in M^{P(X)}$ is called an M-fuzzifying topology on X iff it satisfies the following axioms:

$(1)\mathcal{T}(X)=\mathcal{T}(\phi)=\top,$

$(2)\forall A,B\in P(X),\mathcal{T}(A\cap B)\ge\mathcal{T}(A)\wedge\mathcal{T}(B),$

$(3)\forall\{A_j\mid j\in J\}\subseteq P(X),\mathcal{T}\bigcup_{j\in J}A_j)\ge\bigwedge_{j\in J}\mathcal{T}(A_j).$ The pair $(X,\mathcal{T})$ is called an M-fuzzifying topological space.

Definition 1.8

(2.13). A structure $(L,\vee,\wedge,*,\rightarrow,\perp,\top)$ is called a complete residuated lattice if f

(1) $(L,\vee,\wedge,\perp,\top)$ is a complete lattice whose greatest and least element are $\perp,\top$ respectively,

(2) $(L,*,\top)$ is a commutative monoid, i.e.,

(a) $*$ is a commutative and associative binary operation on L, and

(b) $\forall a,\in L,a*\top=\top*a=a,$

(3)(a) $*$ is isotone,

(b) $\rightarrow$ is a binary operation on L which is antitone in the ...first and isotone in the second variable,

(c) $\rightarrow$ is couple with $*$ as: $a*b\le c$ iff $a\le b\rightarrow c\ \ \forall a,b,c\in L$

2. (2,M)-Double fuzzifying Continuous mapping.

Definition 2.1

Let X be a nonempty set. The pair $(\mathcal{T},\mathcal{T}^*)$ of maps $\mathcal{T},\mathcal{T}^*:2^X\rightarrow M$ is called an (2,M)-double fuzzifying topology on X if it satisfies the following conditions:

(DO1) $\mathcal{T}(A)\le\mathcal{T}^*(A)\rightarrow\perp$, for each $A\in 2^X$,

(DO2 $\mathcal{T}(X)=\mathcal{T}(\varnothing)=\top$ and $\mathcal{T}^*(X)=\mathcal{T}^*(\varnothing)=\perp,$

(DO3) $\mathcal{T}(A\cap B)\ge\mathcal{T}(A)\wedge\mathcal{T}(B)$ and

$\mathcal{T}^*(A\cap B)\le\mathcal{T}^*(A)\vee\mathcal{T}^*(B),$ for each $A,B\in 2^X.$

(DO4) $\mathcal{T}(\bigcup_{i\in\Gamma}A_i)\ge\wedge_{i\in\Gamma}\mathcal{T}(A_i)$ and $\mathcal{T}^*(\bigcup_{i\in\Gamma}A_i)\le\bigvee_{i\in\Gamma}\mathcal{T}^*(A_i),$ for each $\{A_i:i\in\Gamma\}\subseteq 2^X.$

The pair $(X,\mathcal{T},\mathcal{T}^*)$ is called an (2,M)-double fuzzifying topological space. And $\mathcal{T}(A)$ and $\mathcal{T}^*(A)$ may be explained as a gradation of openness and gradation of nonopenness for A.

Remark 2.1

Let $\mathcal{T}:2^X\rightarrow I$ be fuzzifying topology on X. Define a map $\mathcal{T}^*:2^X\rightarrow I$ by $\mathcal{T}^*(A)=\mathcal{T}(A)\rightarrow\perp$. Then when $M=I$, $\odot=\wedge$ and $\oplus=\vee,(\mathcal{T},\mathcal{T}^*)$ is an (2,M)-double fuzzifying topology on X. Therefore, (2,M)-double fuzzifying topology is a generalization of fuzzifying topology due to (13,14) and (15).

Definition 2.2

Let $(X,\mathcal{T}_1,\mathcal{T}_1^*)$ and $(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ be two (2,M)-double fuzzifying topological spaces and for each $B\in 2^Y$. Then, The map $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\rightarrow(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is called an M-double fuzzifying continuous map, if $\wedge_{B\in 2^Y}\mathcal{T}_1(f^{-1}(B))\ge\mathcal{T}_2(B)$ and $\bigvee_{B\in 2^Y}\mathcal{T}_1^*(f^{-1}(B))\le\mathcal{T}_2^*(B)$.

Example 2.1

Let $X=Y=\{a,b\},L=[0,1]$ and if $\mathcal{T}_1,\mathcal{T}_1^*$ and $\mathcal{T}_2,\mathcal{T}_2^*$ defined as follows:

$\mathcal{T}_1(B)=\begin{cases}1 & If\ B\in\{X,\phi\}\\ \frac{1}{3} & If\ B\in\{\{a\},\{b\}\}\end{cases}$, $\mathcal{T}_1^*(B)=\begin{cases}0 & If\ B\in\{X,\phi\}\\ \frac{2}{3} & If\ B\in\{\{a\},\{b\}\}\end{cases}$ and

$\mathcal{T}_2(B)=\begin{cases}1 & If\ B\in\{Y,\phi\}\\ \frac{1}{3} & If\ B\in\{\{a\},\{b\}\}\end{cases}$, $\mathcal{T}_2^*(B)=\begin{cases}0 & If\ B\in\{Y,\phi\}\\ \frac{2}{3} & If\ B\in\{\{a\},\{b\}\}\end{cases}$ The

pairs $\mathcal{T}_1,\mathcal{T}_1^*$ and $\mathcal{T}_2,\mathcal{T}_2^*$ is called an (2,M)-double fuzzifying topological spaces on X. The map $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\rightarrow(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ define by $f(a)=b$ and $f(b)=a$ is an M-double fuzzifying continuous map.

Theorem 2.1

Let $\{(X_i,\mathcal{T}_i,\mathcal{T}_i^*)\}_{i\in\Gamma}$ be a family of an (2,M)-double fuzzifying topological space. And let Y be a set, let $f_i:X_i\rightarrow Y$ be a mapping for each $i\in\Gamma.$ Define a map $\mathcal{T},\mathcal{T}^*:2^Y\rightarrow M$ by

$\mathcal{T}(B)=\bigwedge_{i\in\Gamma}\mathcal{T}_i(f_i^{-1}(B)), \quad \mathcal{T}^*(B)=\bigvee_{i\in\Gamma}\mathcal{T}_i^*(f_i^{-1}(B)).$ For all $B\in 2^Y$. Then:

(1) $(\mathcal{T},\mathcal{T}^*)$ is an (2,M)-double fuzzifying topological space on Y for which each f_i is an M-double fuzzifying continuous mapping.

(2) $f:(Y,\mathcal{T},\mathcal{T}^*)\to(Z,\mathcal{T}_Z,\mathcal{T}_Z^*)$ is an M-double fuzzifying continuous map iff each $f\circ f_i:(X,\mathcal{T}_i,\mathcal{T}_i^*)\to(Z,\mathcal{T}_Z,\mathcal{T}_Z^*)$ is an M-double fuzzifying continuous map.

Proof

(1) From the definition of $(\mathcal{T},\mathcal{T}^*)$ easily get (DO1) and (DO2) are trivial.

(DO3)

$$\mathcal{T}(A\cap B)=\bigwedge_{i\in\Gamma}\mathcal{T}_i(f_i^{-1}(A)\cap(f_i^{-1}(B))$$
$$\geq\left(\bigwedge_{i\in\Gamma}\mathcal{T}_i(f_i^{-1}(A))\wedge(\bigwedge_{i\in\Gamma}\mathcal{T}_i(f_i^{-1}(B)))\right)$$
$$\geq\mathcal{T}(A)\wedge\mathcal{T}(B)$$

and,

$$\mathcal{T}^*(A\cap B)=\left(\bigvee_{i\in\Gamma}\mathcal{T}_i^*(f_i^{-1}(A))\wedge\bigvee_{i\in\Gamma}\mathcal{T}_i^*(f_i^{-1}(B)))\right)$$
$$\leq\left(\bigvee_{i\in\Gamma}\mathcal{T}_i^*(f_i^{-1}(A))\vee(\bigvee_{i\in\Gamma}\mathcal{T}_i^*(f_i^{-1}(B)))\right)$$
$$\leq\mathcal{T}^*(A)\vee\mathcal{T}^*(B).$$

(DO4) For any family $\{A_i\}_{i\in\Gamma}\subseteq 2^Y$

$$\mathcal{T}(\bigcup_{i\in\Gamma}A_i)=\bigwedge_{i\in\Gamma}\mathcal{T}_i(f_i^{-1}(\bigcup_{i\in\Gamma}A_i))$$
$$=\bigwedge_{i\in\Gamma}\mathcal{T}_i(\bigcup_{i\in\Gamma}(f_i^{-1}((A_i))$$
$$\geq\bigwedge_{i\in\Gamma}\bigwedge_{i\in\Gamma}\mathcal{T}_i((f_i^{-1}((A_i))$$
$$=\bigwedge_{i\in\Gamma}\mathcal{T}(A_i),$$

and,

$$\mathcal{T}^*(\bigcup_{i\in\Gamma}A_i)=\bigvee_{i\in\Gamma}\mathcal{T}_i^*(f_i^{-1}(\bigcup_{i\in\Gamma}A_i))$$
$$=\bigvee_{i\in\Gamma}\mathcal{T}_i^*(\bigcup_{i\in\Gamma}(f_i^{-1}((A_i))$$
$$\leq\bigvee_{i\in\Gamma}\bigvee_{i\in\Gamma}\mathcal{T}_i^*((f_i^{-1}((A_i))$$
$$=\bigvee_{i\in\Gamma}\mathcal{T}_i^*(A_i)$$

(2) $(\Leftarrow)$ Since $f\circ f_i:(X,\mathcal{T}_i,\mathcal{T}_i^*)\to(Z,\mathcal{T}_Z,\mathcal{T}_Z^*)$ is an M-double fuzzifying continuous, we have for each $m\in 2^Z,\mathcal{T}_Z(m)\leq\bigwedge_{m\in 2^Z}\mathcal{T}_i((f\circ f_i)^{-1}(m))=\bigwedge_{m\in 2^Z}\mathcal{T}_i((f_i^{-1}((m)).$ an $\mathcal{T}_Z^*(m)\geq\bigvee_{m\in 2^Z}\mathcal{T}_i^*((f\circ f_i)^{-1}(m))=\bigvee_{m\in 2^Z}\mathcal{T}_i^*((f_i^{-1}(f^{-1}(m)).$ From the defination of $(\mathcal{T},\mathcal{T}^*),\mathcal{T}_Z(m)\leq\bigwedge_{m\in 2^Z}\mathcal{T}(f^{-1}(m)),\mathcal{T}_Z^*(m)\geq\bigvee_{m\in 2^Z}\mathcal{T}^*(f^{-1}(m)).$ For all $B\in 2^Z$. Hence $f:(Y,\mathcal{T},\mathcal{T}^*)\to(Z,\mathcal{T}_Z,\mathcal{T}_Z^*)$ is an M-double fuzzifying continuous map.

$(\Rightarrow)$ simple.

Definiation 2.3

Let $(\mathcal{T},\mathcal{T}^*)$ be defined as in Theorem 2.1. Then the frame $(\mathcal{T},\mathcal{T}^*)$ is called final (2,M)-double fuzzifying topology on Y associated with the families $\{(X_i,\mathcal{T}_i,\mathcal{T}_i^*)\}_{i\in\Gamma}$ and $(f_i)_{i\in\Gamma}$.

Corollary 2.1

Let $\{(X_i,\mathcal{T}_i,\mathcal{T}_i^*)\}_{i\in\Gamma}$ be a family of (2,M)-double fuzzifying topological spaces, for $(i\neq j)\in\Gamma$ and $X_i\cap X_j=\phi, X=\bigcup X_i$. Let $id_i:X_i\to X$ be identity map for which $i\in\Gamma$.

Define the map $\mathcal{T},\mathcal{T}^*:2^Y\to M$ by

$(\mathcal{T},B)=\bigwedge_{i\in\Gamma}\mathcal{T}_i(id_i^{-1}(B)),\mathcal{T}_i^*(B)=\bigvee_{i\in\Gamma}\mathcal{T}_i^*(id_i^{-1}(B)).$ For all $B\in 2^Y$

Then:

(1) $(\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space on X for each id_i is an M-double fuzzifying continuous map.

(2) $f:(X,\mathcal{T},\mathcal{T}^*)\to(Z,\mathcal{T}_Z,\mathcal{T}_Z^*)$ is an M-double fuzzifying continuous map iff each $f\circ id_i:(X,\mathcal{T}_i,\mathcal{T}_i^*)\to(Z,\mathcal{T}_Z,\mathcal{T}_Z^*)$ is an M-double fuzzifying continuous map.

Corollary 2.2

Let Y be a set and $(X,\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space, let $f:X\to Y$ be a surjective mapping. Define mappings $\mathcal{T}^f,\mathcal{T}^{*f}:2^Y\to M$

By $\mathcal{T}^{f(B)}=\bigwedge_{i\in\Gamma}\mathcal{T}(f^{-1}(B)),\ \mathcal{T}^{*f}(B)=\bigvee_{i\in\Gamma}\mathcal{T}^*(f^{-1}(B))$ for all $B\in 2^Y$.

Then:

(1) $(\mathcal{T}^f,\mathcal{T}^{*f})$ is an (2,M)-double fuzzifying topological space on X which f is an M-double fuzzifying continuous map.

(2) $g:(Y,\mathcal{T}^f,\mathcal{T}^{*f})\to(Z,\mathcal{T}_Z,\mathcal{T}_Z^*)$ is an M-double fuzzifying continuous map iff each $g\circ f:(X,\mathcal{T},\mathcal{T}^*)\to(Z,\mathcal{T}_Z,\mathcal{T}_Z^*)$ is an M-double fuzzifying continuous map.

Definiation 2.4

Let $(X,\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space. And Y a set, let $f:X\to Y$ be a surjective mapping. The (2;M)-double fuzzifying topological space $\mathcal{T}^f$ on Y associated the $(X,\mathcal{T},\mathcal{T}^*)$ and f is called the quotient (2;M)-double fuzzifying topological space and the map is called M-double fuzzifying quotient map.

Definition 2.5

Let $(X,\mathcal{T}_1,\mathcal{T}_1^*)$ and $(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ be two (2,M)-double fuzzifying topological spaces and for each $B\in 2^Y$. Then,

(i) The map $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is called an M-double fuzzifying openess, if $\mathcal{T}_1(f^{-1}(B))\leq\bigwedge_{B\in 2^X}\mathcal{T}_2(f(f^{-1}(B)))$ and $\mathcal{T}_1^*(B)\geq\bigvee_{B\in 2^X}\mathcal{T}_2^*(f(f^{-1}(B))),$

(ii) The map $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is called an M-double fuzzifying closness if $\mathcal{T}_1(f^{-1}(B))\to\perp\leq(\bigvee_{B\in 2^X}\mathcal{T}_2(f(f^{-1}(B))))\to\perp$ and $\mathcal{T}_1^*(f^{-1}(B))\to\perp\geq(\bigwedge_{B\in 2^X}\mathcal{T}_2^*(f(f^{-1}(B))))\to\perp$.

Theorem 2.2

Let Let $(X,\mathcal{T}_1,\mathcal{T}_1^*)$ and $(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ be two (2,M)-double fuzzifying topological spaces, let $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ be a surjective an M-double fuzzifying continuous mapping. Then

(1) If $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is an M-double fuzzifying openess, then f is M-double fuzzifying quotient map.

(2) $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is an M-double fuzzifying closness, then f is M-double fuzzifying quotient map.

Proof

(1) Only, should prove that $\mathcal{T}_2=\mathcal{T}^f$. So, From Corollary 2.2 and

Definition 2.5 we have, $\mathcal{T}_2(B) \le \mathcal{T}^f(B)$, $\mathcal{T}_2^*(B) \ge \mathcal{T}^{*f}(B)$ for all $B \in 2^Y$ Conversely, we have

$$\mathcal{T}^f(B) = \bigwedge_{i\in\Gamma} \mathcal{T}_1(f^{-1}(B)) \le \bigwedge_{i\in\Gamma}\left(\wedge_{B\in 2^X} \mathcal{T}_2(f(f^{-1}(B)))\right) = \bigwedge_{i\in\Gamma}\left(\wedge_{B\in 2^X} \mathcal{T}_2(B)\right) = \mathcal{T}_2(B),$$

$$\mathcal{T}^{*f}(B) = \bigvee_{i\in\Gamma} \mathcal{T}_1^*(f^{-1}(B)) \ge \bigvee_{i\in\Gamma}\left(\vee_{B\in 2^X} \mathcal{T}_2^*(f(f^{-1}(B)))\right) = \bigvee_{i\in\Gamma}\left(\vee_{B\in 2^X} \mathcal{T}_2^*(B)\right) = \mathcal{T}_2((B).$$

(3) Trivial.

Theorem 2.3.

A map $c: 2^X \times M_0 \times M_1 \to 2^X$ is called an M- fuzzifying closure operator if for each $A, B \in 2^X$, $r \in L_0$, $s \in L_1$ with $r \le s \to \perp$. The operator c satisfies the following conditions:

$c(1)$ $c(\phi, r, s) = \phi$,

$c(2)$ $A \subseteq c(A, r, s)$,

$c(3)$ *If* $A \subseteq B$, *then* $c(A, r, s) \subseteq c(B, r, s)$,

$c(4)$ *If* $r \le r'$ *and* $s \ge s'$ *with* $r' \le s' \to \perp$ *then* $c(A, r, s) \subseteq c(A, r', s')$,

$c(5)$ $c(A_1 \cup A_2, r \odot r', s \oplus s') = c(A_1, r, s) \cup c(A_2, r', s')$, Then the pair (X, c) is an M- fuzzifying closure space. An M- fuzzifying closure space (X, c) is called topological if

$c(6)$ $c(c(A, r, s)) = c(A, r, s)$ for each $A, B \in 2^X$, $r \in L_0$, $s \in M_1$ with $r \le s \to \perp$.

Definiation 2.6

Let (X, c_1) and (Y, c_2) be two M-fuzzifying closure spaces. A map $f: (X, c_1) \to (Y, c_2)$ is said to be a c-map if for all $A \in 2^X$, $r \in M_0, s \in M_1$ with $r \to s\ \perp$, $f(c_1(A, r, s)) \le c_2(f(A), r, s))$.

Theorem 2.4

Let Y be a set and let $\{(X, c_i)\}_{i\in\Gamma}$ be a collection of an M-fuzzifying closure spaces, let $f_i: X_i \to Y$ be a surjective mapping for each $i \in \Gamma$. Define a mappings

$c: 2^Y \times M_0 \times M_1 \to 2^Y$ by $c_i(A, r, s) = \bigvee_{i\in\Gamma} f_i(c_i(f_i^{-1}(A), r, s))$.

Then:

(1) c is an M- fuzzifying space on Y for each fi is c-map,

(2) $f: (Y, c) \to (Z, c_Z)$ is C-map iff each $f \circ f_i: (X, c_i) \to (Z, c_Z)$ is c-map.

Proof

(1) $c(1), c(3), c(4)$ *and* $c(5)$ come directly from the definition of c. For $c(2)$, we have,

$$c_i(A, r, s) = \bigwedge_{i\in\Gamma} \mathcal{T}_i(f_i^{-1}(A) \cap (f_i^{-1}(B)))$$

$$\ge f_i(c_i(f_i^{-1}(A), r, s))$$

$$\ge c_i(f_i(f_i^{-1}(A), r, s)) \ge A$$

Hence $f_i: (X_i, c) \to (Y, c)$ is c-map.

(2) $(\Longrightarrow)$ simple.

$(\Longleftarrow)$ Let $f \circ f_i: (X, c_i) \to (Z, c_Z)$ be a c-map, we have $f \circ f_i(c_i(A_i, r, s)) \le c_Z(f \circ f_i(A_i), r, s))$

It implies

$$f(c(A, r, s)) = f \bigvee_{i\in\Gamma} f_i(c_i(f_i^{-1}(A), r, s)))$$

$$= \bigvee_{i\in\Gamma} f(f_i(c_i(f_i^{-1}(A), r, s)))$$

$$\le c_Z(f \circ f_i(f_i^{-1}(A)), r, s$$

$$= c_Z(f(A), r, s).$$

From Theorem 2.4 we introduce the following definition

Definiation 2.7

The structure c is called an M- fuzzifying operator on Y associated with the families $\{(X_i, c_i)\}_{i\in\Gamma}$ *and* $(f_i)_{i\in\Gamma}$.

Corollary 2.3

Let $\{(X_i, c_i)\}_{i\in\Gamma}$ be a family of an M- fuzzifying operator, for $(i \ne j) \in \Gamma$ and $X_i \cap X_j = \phi$, $X = \bigcup_{i\in\Gamma} X_i$. Let $id_i: X_i \to X$ be identity map for which $i \in \Gamma$.

Define the map $c: 2^X \times M_0 \times M_1 \to 2^X$ by

$c(A, r, s) = \bigvee_{i\in\Gamma} id_i(c_i(id_i^{-1}(A), r, s))$.

Then:

(1) c is an M- fuzzifying operator on X for which *idi* is c-map,

(2) $f: (Y, c) \to (Z, c_Z)$ is c-map iff each $f \circ id_i: (X, c_i) \to (Z, c_Z)$ is c-map.

Definiation 2.8

Let (X, c) be an M- fuzzifying operator. And Y a set, let $f: X \to Y$ be a surjective map. Define the map $c: 2^Y \times M_0 \times M_1 \to 2^Y$ by $c^f(A, r, s) = f(c_i(f^{-1}(A), r, s))$. Then (Y, c^f) induced by f is called an M- fuzzifying quotient space of (X, c) and the function f is called an an M-fuzzifying quotient map.

Theorem 2.5

Let Y be a set and $\{(X_i, \mathcal{T}_i, \mathcal{T}_i^*)\}_{i\in\Gamma}$ be a collection of an (2,M)- fuzzifying topological spaces, let $f_i: X_i \to Y$, be a surjective map for each $i \in \Gamma$ and

$\{(X_i, c_{\mathcal{T}_i, \mathcal{T}_i^*})\}_{i\in\Gamma}$ a collection of an M- fuzzifying operator induced by $\{(X_i, \mathcal{T}_i, \mathcal{T}_i^*)\}_{i\in\Gamma}$. Define the functions $\mathcal{T}_i$ and $\mathcal{T}_c$ on Y by $\bigwedge_{B\in 2^Y} \mathcal{T}_i(f^{-1}(B)) \ge \mathcal{T}_c(B)$ and $\bigvee_{B\in 2^Y} \mathcal{T}_i^*(f^{-1}(B)) \le \mathcal{T}_c^*(B)$ and the map $c: 2^Y \times M_0 \times M_1 \to 2^Y$ by $c(A, r, s) = \bigvee_{i\in\Gamma} f_i(c_{\mathcal{T}_i, \mathcal{T}_i^*}(f^{-1}(A), r, s))$. Then $f_i: (X_i, \mathcal{T}_i, \mathcal{T}_i^*) \to (Y, \mathcal{T}_c, \mathcal{T}_c^*)$ an M- fuzzifying continuous mapping.

Proof

Suppose there exists $B \in 2^Y$ such that $\bigwedge_{B\in 2^Y} \mathcal{T}_i(f^{-1}(B)) \not\ge \mathcal{T}_c(B)$ and $\bigvee_{B\in 2^Y} \mathcal{T}_i^*(f^{-1}(B)) \not\le \mathcal{T}_c^*(B)$ then there exists $r_0 \in L_0, s_0 \in L_1$ with $c(X - B, r, s) = \bigvee_{i\in\Gamma} f_i(c_{\mathcal{T}_i, \mathcal{T}_i^*}(f_i^{-1}(X - B), r, s)) = X - B$ such that $\mathcal{T}_c(B) \le r_0 < \bigwedge_{B\in 2^Y} \mathcal{T}_i(f^{-1}(B))$ and $\mathcal{T}_c^*(B) \ge s_0 > \bigvee_{B\in 2^Y} \mathcal{T}_i^*(f^{-1}(B))$.

On the other hand, we have

$$\mathcal{T}_c(X - B) = c(X - B, r, s) = \bigvee_{i\in\Gamma} f_i(c_{\mathcal{T}_i, \mathcal{T}_i^*}(f_i^{-1}(X - B), r, s))$$

$$\ge f_i(c_{\mathcal{T}_i, \mathcal{T}_i^*}(X - f_i^{-1}(B)), r, s.$$

It implies

$$\mathcal{T}_c(f_i^{-1}(B)) = \mathcal{T}_c(X - f_i^{-1}(B)) = c(X - f_i^{-1}(B), r, s) \ge f_i(c_{\mathcal{T}_i, \mathcal{T}_i^*}(X - f_i^{-1}(B)), r_0, s_0)) \ge c_{\mathcal{T}_i, \mathcal{T}_i^*}(X - f_i^{-1}(A)), r_0, s_0) = f_i^{-1}(B).$$ Then

$\mathcal{T}_c(f_i^{-1}(B)) \ge \bigwedge_{B\in 2^Y} \mathcal{T}_i(f_i^{-1}(B))$ and by the same ways $\mathcal{T}_c^*(B) \le \bigvee_{B\in 2^Y} \mathcal{T}_i^*(f^{-1}(B))$, we have $\mathcal{T}_i(f_i^{-1}(B)) \le r_0$ *and* $\mathcal{T}_i^*((f_i^{-1}(B)) \ge s_0$, which a contradiction.

Hence $f_i: (X_i, \mathcal{T}_i, \mathcal{T}_i^*) \to (Y, \mathcal{T}_c, \mathcal{T}_c^*)$ an M-double fuzzifying continuous mapping.

3. Totally Continuous in (2;M)-double fuzzifying topological spaces.

Definition 3.1

Let $(X,\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space. Define an (2,M)-double fuzzifying Interior operator $I_{\mathcal{T},\mathcal{T}^*}:2^X\to L^X$ by:

$$I_{\mathcal{T},\mathcal{T}^*}(A)(x)=\bigwedge_{x\in X}(\bigvee_{x\in X}((\bigvee_{x\in B\subseteq A}\mathcal{T}(B))),\bigwedge_{x\in X}(\bigvee_{x\in B\subseteq X-A}(\mathcal{T}^*(B)))).$$

Definition 3.2

Let $(X,\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space. Define an (2,M)-double fuzzifying closure operator $C_{\mathcal{T},\mathcal{T}^*}:2^X\to L^X$ by:

$$c_{\mathcal{T},\mathcal{T}^*}(A)(x)=\bigwedge_{x\in X}(\bigvee_{x\in X}((\bigvee_{x\in B\subseteq X-A}\mathcal{T}(B))\to\perp)),\bigwedge_{x\in X}((\bigvee_{x\in B\subseteq X-A}(\mathcal{T}^*(B))\to\perp)).$$

Example 3.1

Let $X=\{a,b,c\},L=[0,1]$ and. Let $(X,\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space defined by

$$\mathcal{T}(B)=\begin{cases}1 & If\ B\in\{X,\phi,\{a\}\}\\ \frac{3}{4} & If\ \ B\in\{\{c\},\{a,c\}\}\\ 0 & If\ o.w.\end{cases},\quad \mathcal{T}^*(B)=\begin{cases}0 & If\ B\in\{X,\phi,\{a\}\}\\ \frac{1}{4} & If\ \ B\in\{\{c\},\{a,c\}\}\\ 0 & If\ o.w.\end{cases}.$$

If $A=\{a,b\}$. Then $C_{\mathcal{T}_1,\mathcal{T}_1^*}(A)(x)=(1,1)$ and $I_{\mathcal{T}_1,\mathcal{T}_1^*}(A)(x)=(0,0).$

Definition 3.3

Let $(X,\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space. Let $C_{\mathcal{T},\mathcal{T}^*}:2^X\to L^X.$ If its extension $\overline{C}_{\mathcal{T},\mathcal{T}^*}:L^X\to L^X,\ \overline{C}_{\mathcal{T},\mathcal{T}^*}(\lambda)=\bigcup_{\alpha\in L}(\alpha\wedge C_{\mathcal{T},\mathcal{T}^*}(A_\alpha)(x)).\ \lambda\in L^X.$

Where $A_\alpha=\{x:(A)(x)\geq\alpha\}$ satisfies the following statments:

(1) $\overline{C}_{\mathcal{T},\mathcal{T}^*}(\underline{0})=\underline{0},$

(2) $\lambda\leq\overline{C}_{\mathcal{T},\mathcal{T}^*}(\lambda),$

(3) $\overline{C}_{\mathcal{T},\mathcal{T}^*}(\lambda\vee\mu)=\overline{C}_{\mathcal{T},\mathcal{T}^*}(\lambda)\vee\overline{C}_{\mathcal{T},\mathcal{T}^*}(\mu),\ \lambda,\mu\in L^X$

(4) $\overline{C}_{\mathcal{T},\mathcal{T}^*}(\overline{C}_{\mathcal{T},\mathcal{T}^*}(\lambda))\subseteq\overline{C}_{\mathcal{T},\mathcal{T}^*}(\lambda),\lambda\in L^X.$

Definition 3.4

Let $(X,\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space and $A\subseteq X.$

(i) An M-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$ semi open set (briefly, $S_{\mathcal{T},\mathcal{T}^*}(A))$ of $A,$ defined as follows: $S_{\mathcal{T},\mathcal{T}^*}(A,)(x)=\bigwedge_{x\in A}(\overline{C}_{\mathcal{T},\mathcal{T}^*}(\overline{I}_{\mathcal{T},\mathcal{T}^*}(A)(x)),$

(ii) An M-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$ semi closed set (briefly, $SC_{\mathcal{T},\mathcal{T}^*}(A))$ of $A,$ if $S_{\mathcal{T},\mathcal{T}^*}(A)(x)\to\perp$ is An M-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$semi open,

(iii) (iii) AnM-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$ semi clopen set (briefly, $SCL_{\mathcal{T},\mathcal{T}^*}(A))$ of $A,$ if A has $S_{\mathcal{T},\mathcal{T}^*}(A)$ and $SC_{\mathcal{T},\mathcal{T}^*}(A).$

(iv) An M-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$ pre open set (briefly, $P_{\mathcal{T},\mathcal{T}^*}(A))$ of $A,$ defined as follows:

$$P_{\mathcal{T},\mathcal{T}^*}(A)(x)=\bigwedge_{x\in A}(\overline{I}_{\mathcal{T},\mathcal{T}^*}(C_{\mathcal{T},\mathcal{T}^*}(A)(x)),$$

(v) An M-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$ pre closed set (briefly, $PC_{\mathcal{T},\mathcal{T}^*}(A))$ of $A,$ if $P_{\mathcal{T},\mathcal{T}^*}(A)(x)\to\perp$ is An M-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$ pre open.

(vi) AnM-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$ pre clopen set (briefly, $PCL_{\mathcal{T},\mathcal{T}^*}(A))$ of $A,$ if A has $P_{\mathcal{T},\mathcal{T}^*}(A)$ and $PC_{\mathcal{T},\mathcal{T}^*}(A).$

Remark 3.1

An M-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$clopen set (briefly, $CL_{\mathcal{T},\mathcal{T}^*}(A))$ of $A,$ if and only if A has $PC_{\mathcal{T},\mathcal{T}^*}(A)$ and $SCL_{\mathcal{T},\mathcal{T}^*}(A).$

Example 3.2

In Example 3.1 $\bigwedge_{x\in A}(\overline{C}_{\mathcal{T},\mathcal{T}^*}(I_{\mathcal{T},\mathcal{T}^*}(A)(x))=(0,0).$

Definition 3.5

Let $(X,\mathcal{T}_1,\mathcal{T}_1^*)$ and $(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ be two (2,M)-double fuzzifying topological spaces.

Then,

(i) The map $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is called an M-double fuzzifying totally continuous (briefly, *dftc*), if for each $B\in 2^Y$ have $C_{\mathcal{T},\mathcal{T}^*}(B)=I_{\mathcal{T},\mathcal{T}^*}(B)=B,$ then,

$\bigwedge_{B\in 2^Y}\mathcal{T}_1(f^{-1}(B))\geq\mathcal{T}_2(B)$ and

$\bigvee_{B\in 2^Y}\mathcal{T}_1^*(f^{-1}(B))\leq\mathcal{T}_2^*(B).$

(ii) The $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is called an M-double fuzzifying semi continuous (briefly, *df sc*), if for each $S_{\mathcal{T},\mathcal{T}^*}(B))$ of $B\in 2^Y$, $\bigwedge_{B\in 2^Y}\mathcal{T}_1(f^{-1}(B))\geq\mathcal{T}_2(B)$ and $\bigvee_{B\in 2^Y}\mathcal{T}_1^*(f^{-1}(B))\leq\mathcal{T}_2^*(B).$

(iii) The map $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is called an M-double fuzzifying totally semicontinuous (briefly, *dftsc*), if for each $SCL_{\mathcal{T},\mathcal{T}^*}(B)$ of $B\in 2^Y$, $\bigwedge_{B\in 2^Y}\mathcal{T}_1(f^{-1}(B))\geq\mathcal{T}_2(B)$ and $\bigvee_{B\in 2^Y}\mathcal{T}_1^*(f^{-1}(B))\leq\mathcal{T}_2^*(B).$

(iv) The map $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is called an M-double fuzzifying totally precontinuous (briefly, *dftpc*), if for each $PCL_{\mathcal{T},\mathcal{T}^*}(B)$ of $B\in 2^Y$,

$\bigwedge_{B\in 2^Y}\mathcal{T}_1(f^{-1}(B))\geq\mathcal{T}_2(B)$ and $\bigvee_{B\in 2^Y}\mathcal{T}_1^*(f^{-1}(B))\leq\mathcal{T}_2^*(B).$

Definition 3.6

Let $(X,\mathcal{T}_1,\mathcal{T}_1^*)$ and $(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ be two (2;M)-double fuzzifying topological spaces and for each $B\in 2^Y$. Then,

(i) The map $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is called an M-double fuzzifying openess, if

$(\mathcal{T}_1(B))\leq\bigwedge_{B\in 2^X}\mathcal{T}_2(f(B))$ and $\mathcal{T}_1^*(B)\geq\bigvee_{B\in 2^X}\mathcal{T}_2^*(f(B)),$

(ii) The map $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is called an M-double fuzzifying closness, $\mathcal{T}_1(B)\to\perp\leq(\bigvee_{B\in 2^X}(\mathcal{T}_2(f(B)))\to\perp$ and $\mathcal{T}_1^*(B)\to\perp\geq\bigwedge_{B\in 2^X}\mathcal{T}_2^*(f(B)))\to\perp.$

Theorem 3.1

Let $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ be a mapping. Then the following are equivalent:

(i) f is a *dftc* mapping,

(ii) $f^{-1}(B)$ is an $CL_{\mathcal{T},\mathcal{T}^*}(B))$ of $B.$ such that

$\mathcal{T}_2(B)\to\perp\geq\bigwedge_{B\in 2^X}(\mathcal{T}_1(f^{-1}(B)))$ and

$\bigvee_{B\in 2^Y}\mathcal{T}_1^*(f^{-1}(B))\geq\mathcal{T}_2^*(B)\to\perp$ for each $B\in 2^Y$,

(iii) $\overline{C}_{\mathcal{T},\mathcal{T}^*}(\mathcal{T}_2(B))\to\perp\geq\bigwedge_{B\in 2^X}(\overline{C}_{\mathcal{T},\mathcal{T}^*}(\mathcal{T}_1(f^{-1}(B)))$ and $\bigvee_{B\in 2^Y}\overline{C}_{\mathcal{T},\mathcal{T}^*}(\mathcal{T}_1^*(f^{-1}(B)))\geq\overline{C}_{\mathcal{T},\mathcal{T}^*}(\mathcal{T}_2^*(B))\to\perp$ for each $B\in 2^Y$,

(iv) $\overline{I}_{\mathcal{T},\mathcal{T}^*}(\mathcal{T}_2(B))\to\perp\geq\bigwedge_{B\in 2^X}(\overline{I}_{\mathcal{T},\mathcal{T}^*}(\mathcal{T}_1(f^{-1}(B)))$ and $\bigvee_{B\in 2^Y}\overline{I}_{\mathcal{T},\mathcal{T}^*}(\mathcal{T}_1^*(f^{-1}(B)))\geq\overline{I}_{\mathcal{T},\mathcal{T}^*}(\mathcal{T}_2^*(B))\to\perp$ for each $B\in 2^Y.$

Proof

Follow directly from Definition 3.3 and Definition 3.5

Definition 3.6

Let $(X,\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space and $A \subseteq X$.

(i) An M-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$generalized closed set $GC_{\mathcal{T},\mathcal{T}^*}(A)$ of A, defined as follows:

$$GC_{\mathcal{T},\mathcal{T}^*}(A)(x) = \bigwedge_{x\in A}(C_{\mathcal{T},\mathcal{T}^*}(A)(x),$$

(ii) An M-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$ generalized open set $GO_{\mathcal{T},\mathcal{T}^*}(A)$ of A, if $GO_{\mathcal{T},\mathcal{T}^*}(A)(x) \to \perp$ is an M-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$generalized closed.

Definition 7.7

Let $(X,\mathcal{T}_1,\mathcal{T}_1^*)$ and $(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ be two (2,M)-double fuzzifying topological spaces. Then,

(i) The map $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is called double fuzzifying irresolute if, An M-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$semi open set of $\bigwedge_{B\in 2^Y}(\mathcal{T}_2(B)) \le (C_{\mathcal{T},\mathcal{T}^*}(I_{\mathcal{T},\mathcal{T}^*}(\mathcal{T}_1(f^{-1}(B))))(x)$ and $\bigvee_{B\in 2^Y}(\mathcal{T}_2^*(B) \ge (C_{\mathcal{T},\mathcal{T}^*}(I_{\mathcal{T},\mathcal{T}^*}(\mathcal{T}_1^*(f^{-1}(B))))(x)$,

(ii) The map $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is called double fuzzifying pre semi close, if $\bigwedge_{B\in 2^X}((C_{\mathcal{T},\mathcal{T}^*}(I_{\mathcal{T},\mathcal{T}^*}(\mathcal{T}_1(f^{-1}(B))))(x))\to\perp \le (\mathcal{T}_2((C_{\mathcal{T},\mathcal{T}^*}(I_{\mathcal{T},\mathcal{T}^*}(\mathcal{T}_1(f(B))))(x)))\to\perp)$ and $\bigvee_{B\in 2^X}((C_{\mathcal{T},\mathcal{T}^*}(I_{\mathcal{T},\mathcal{T}^*}(\mathcal{T}_1^*(B)\to\perp) \ge ((C_{\mathcal{T},\mathcal{T}^*}(I_{\mathcal{T},\mathcal{T}^*}(\mathcal{T}_2^*(f(B))\to\perp)$,

Definition 3.8 Let $(X,\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space and $A \subseteq X$.

(i) An M-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$semi generalized closed set $SGC_{\mathcal{T},\mathcal{T}^*}(A)$ of A, defined as follows: $\bigwedge_{x\in A}(C_{\mathcal{T},\mathcal{T}^*}(A)(x)) \le \bigwedge_{x\in A}(C_{\mathcal{T},\mathcal{T}^*}(I_{\mathcal{T},\mathcal{T}^*}(A)(x)))$,

(ii) An M-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$semi generalized open set $SGO_{\mathcal{T},\mathcal{T}^*}(A)$ of A, if $SGO_{\mathcal{T},\mathcal{T}^*}(A)(x)\to\perp$ is an M-double fuzzifying $(\mathcal{T},\mathcal{T}^*)-$semi generalized closed,

Theorem 2.1

Let $(X,\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space.

(1) Let $A \subseteq X$, A has an $SGC_{\mathcal{T},\mathcal{T}^*}(A)$ then B has an $SGC_{\mathcal{T},\mathcal{T}^*}(B)$.

(2) If A has an $SC_{\mathcal{T},\mathcal{T}^*}(A)$, then it has $SGC_{\mathcal{T},\mathcal{T}^*}(A)$.

Proof

Follow directly from Definition 3.3 and Definition 3.8

Remark 3.1

The converse of (2) in Theorem 2.1 is not true in general.

Remark 3.2

Let $(X,\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space. Let $A \subseteq B$, then the concepts of $SC_{\mathcal{T},\mathcal{T}^*}(A)$, and $SGC_{\mathcal{T},\mathcal{T}^*}(A)$ are independent concepts.

Theorem 3.2

Let $(X,\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space. Define the an operator semi generalized M-double fuzzifying closure operator $SGC_{\mathcal{T},\mathcal{T}^*}: 2^X \to L^X$ by :

$$SGC_{\mathcal{T},\mathcal{T}^*}(A)(x) = \bigwedge_{x\in X}(\bigvee_{x\in X}((\bigvee_{x\in B\subseteq X-A}\mathcal{T}(B))\to\perp),$$

$\bigwedge_{x\in X}(\bigvee_{x\in B\subseteq X-A}\mathcal{T}^*(B))\to\perp))$. Such that B has an $SGC_{\mathcal{T},\mathcal{T}^*}(B)$, the operator $SGC_{\mathcal{T},\mathcal{T}^*}$, satisfies the following statments:

(1) $SGC_{\mathcal{T},\mathcal{T}^*}(\phi) = 0$,

(2) $\bigwedge_{x\in X}(\mathcal{T}(A),\mathcal{T}^*(A)) \le SGC_{\mathcal{T},\mathcal{T}^*}(A)$,

(3) $SGC_{\mathcal{T},\mathcal{T}^*}(A\cup B) \ge (SGC_{\mathcal{T},\mathcal{T}^*}(A)) \vee (SGC_{\mathcal{T},\mathcal{T}^*}(B))$,

(4) $SGC_{\mathcal{T},\mathcal{T}^*}(SGC_{\mathcal{T},\mathcal{T}^*}(A)) = SGC_{\mathcal{T},\mathcal{T}^*}(A)$

(5) If A has an $SGC_{\mathcal{T},\mathcal{T}^*}(A)$ then it has $SGC_{\mathcal{T},\mathcal{T}^*}(A) = \bigwedge_{x\in X}(\mathcal{T}(A),\mathcal{T}^*(A))$,

(6) $\bigwedge_{x\in A}SGC_{\mathcal{T},\mathcal{T}^*}(A)(x) \le \bigwedge_{x\in A}GC_{\mathcal{T},\mathcal{T}^*}(A)(x) \le \bigwedge_{x\in A}C_{\mathcal{T},\mathcal{T}^*}(A)(x)$.

Proof

Follow directly from Definition 3.3 and Definition 3.8

Theorem 3.3

Let $(X,\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space. Define the an operator semi generalized M-double fuzzifying interior operator $SGI_{\mathcal{T},\mathcal{T}^*}: 2^X \to L^X$ by:

$$SGI_{\mathcal{T},\mathcal{T}^*}(A)(x) = \bigwedge_{x\in X}(\bigvee_{x\in X}((\bigvee_{x\in B\subseteq X-A}\mathcal{T}(B)),\ \bigwedge_{x\in X}(\bigvee_{x\in B\subseteq X-A}\mathcal{T}^*(B))).$$

Such that B has an $SGI_{\mathcal{T},\mathcal{T}^*}(B)$. The operator

$$SGI_{\mathcal{T},\mathcal{T}^*}(X-A)(x) = SGI_{\mathcal{T},\mathcal{T}^*}(A)(x)\to\perp.$$

Theorem 2.4

Let $(X,\mathcal{T}_1,\mathcal{T}_1^*)$ and $(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ be two (2,M)-double fuzzifying topological spaces. Then the map $f:(X,\mathcal{T}_1,\mathcal{T}_1^*)\to(Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is called

(i) $df\ ap-$irresolute if $f^{-1}(B)$ has an $SC_{\mathcal{T}_1,\mathcal{T}_1^*}(B)$ for each $B\subseteq Y$ has an $SC_{\mathcal{T}_2,\mathcal{T}_2^*}(B)$,

(ii) $df\ ap-$ semi closed if $f(A)$ has an $SC_{\mathcal{T}_1,\mathcal{T}_1^*}(A)$ for each $A\subseteq Y$ has an $SC_{\mathcal{T}_2,\mathcal{T}_2^*}(A)$.

Definition 3.10

Let $(X,\mathcal{T},\mathcal{T}^*)$ be an (2,M)-double fuzzifying topological space. A set A is called double fuzzifying semi clopen (for short, $df\ clo-set$), if it has an $SC_{\mathcal{T},\mathcal{T}^*}(A)$ and $SC_{\mathcal{T},\mathcal{T}^*}(A)$ for each $A\subseteq X$.

4. Characteraizations of (2,M)-double fuzzifying topology

In this section M is assumed to be a completely distributive complete residuated lattice, where M satisfies the double negation law. In (Corollary 2.15 (Höhle) (2)) proved that the M-fuzzy contiguity relations and (2,M)-fuzzifying topologies are equivalent notions if L is a completely distributive complete MV-algebra. In the following we prove that M-double-fuzzy contiguity relations and (2,M)-double-fuzzifying topology are equivalent notions just if L is a completely distributive complete residuated lattice satisfies the double negation law so that we give a generalization of U. Höhle's result. In (Höhle (2)) the concepts of (2,M)- fuzzifying topology and (2,M)- fuzzifying neighborhood system are equivalent notions. Then our generalization of U. Höhle's result is obtained if we prove that,

(1) (2,M)-double fuzzifying topology and (2,M)- double fuzzifying neighborhood system,

(2) M-double fuzzifying contiguity relation and (2,M)-double fuzzifying neighborhood system,

(3) M-double fuzzifying interior operator and (2,M)-double fuzzifying neighborhood system are equivalent notions.

Definition 4.1

Let X be a nonempty set and $x\in X$. If L satisfies a completely distributive law. Then the pair $(N_x, N_x^*)\in L^{2X}$ is called an (2,M)-double fuzzifying neighborhood system of x if satisfies the following conditions:

$(DN-f_1) N_x(A) \le N_x^*(A) \to \perp$, for each $A \in 2^X$. And

$N_x(X) = N_x(\phi) = \top$, $N_x^*(X) = N_x^*(\phi) = \perp \ \forall x \in X$, (Boundary conditions)

$(DN-f_2) N_x(A \cap B) \ge N_x(A) \wedge N_x(B)$, and

$N_x^*(A \cap B \le N_x^*(A) \vee N_x^*(B)$ for each $A, B \in 2^X$ (Intersection property)

$(DN-u_3) N_x(A) = \top$, $N_x^*(A) = \perp$ whenever $x \notin A$ $(DN-u_4)$ For each $x \in A$, $\forall B \in P(X)$,

$N_x(A) \le \bigvee_{y \notin B} (N_y(A) \vee N_x(B))$ and

$N_x^*(A) \ge \bigwedge_{y \notin B} (N_y^*(A) \wedge N_x^*(B))$.

Theorem 4.1

Let the pair (N_x, N_x^*) be an (2,M)-double fuzzifying neighborhood system. And $(X, \mathcal{T}, \mathcal{T}^*)$ be an $(2, L)-$double fuzzifying topological space. We define the maps $(\mathcal{T}_{(N_x)}, \mathcal{T}^*_{(N_x^*)}) : L^X \to L$ as follows:

$$\mathcal{T}_{(N_x)}(A) = \bigvee_{x \in A} N_x(A),\ \mathcal{T}^*_{(N_x^*)}(A) = \bigwedge_{x \in A} N_x^*(A)$$

Then the pair $(\mathcal{T}_{(N_x)}, \mathcal{T}^*_{(N_x^*)})$ is an $(2, L)$-double fuzzifying topological space induces by (2,M)-double fuzzifying neighborhood system (N_x, N_x^*). Let $(\mathcal{T}, \mathcal{T}^*)$ be an $(2, L)-$double fuzzifying topological space. We define the maps $(N_x)_{\mathcal{T}}$, $(N_x^*)_{\mathcal{T}^*} : L^X \to L$ as follows:

$$(N_x)_{\mathcal{T}}(A) = \mathcal{T}(A),\ (N_x^*)_{\mathcal{T}^*}(A) = \mathcal{T}^*(A) \to \perp$$

Then $((N_x)_{\mathcal{T}}, (N_X^*)_{\mathcal{T}^*})$ is an an (2,M)-double fuzzifying neighborhood system induces by an $(2, L)$-double fuzzifying topological space on X. Furthermore $(\mathcal{T}_{(N_x)_{\mathcal{T}}}, \mathcal{T}^*_{(N_x^*)_{\mathcal{T}^*}}) = (\mathcal{T}, \mathcal{T}^*)$.

Proof

(A) (DO1) For each $A \in L^X$,

$$\begin{aligned}\mathcal{T}^*_{(N_x^*)}(A) \to \perp &= (\bigwedge_{x \in A} (N_x^*(A)) \to \perp \\ &= \bigvee_{x \in A} (N_x^*(A) \to \perp) \\ &\ge \bigvee_{x \in A} N_x(A) \\ &= \mathcal{T}_{(N_x)}(A).\end{aligned}$$

(DO2) $\mathcal{T}_{(N_x)}(X) = \bigwedge_{x \in A} N_x(X) = \top$, $\mathcal{T}_{(N_x)}(\phi) = \bigwedge_{x \in A} N_x(\phi) = \top$ and

$$\mathcal{T}^*_{(N_x^*)}(X) = \bigvee_{x \in A} (N_x^*(X) = \perp,\ \mathcal{T}^*_{(N_x^*)}(\phi) = \bigvee_{x \in A} N_x^*(\phi) = \perp$$

(DO3) for each $A, B \in 2^X$,

$$\begin{aligned}\mathcal{T}_{(N_x)}(A \cap B) &= \bigvee_{x \in A \cap B} N_x(A \cap B) \\ &\ge \bigvee_{x \in A \cap B} (N_x(A) \wedge N_x(B)) \\ &= (\bigvee_{x \in A} \mathcal{T}(A)) \wedge (\bigvee_{x \in B} \mathcal{T}(B)) \\ &= \mathcal{T}_{(N_x)}(A) \wedge \mathcal{T}_{(N_x)}(B).\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}^*_{(N_x^*)}(A \cap B) &= \bigwedge_{x \in A \cap B} N_x^*(A \cap B) \\ &\le \bigwedge_{x \in A \cap B} (N_x^*(A) \wedge N_x^*(B)) \\ &= (\bigwedge_{x \in A} N_x^*(A)) \wedge (\bigwedge_{x \in B} N_x^*(B)) \\ &= \mathcal{T}^*_{(N_x^*)}(A) \wedge \mathcal{T}^*_{(N_x^*)}(B).\end{aligned}$$

(DO4) For each $\{A_i : i \in \Gamma\} \subseteq 2^X$.

$$\begin{aligned}\mathcal{T}_{(N_x)}(\bigcup_{i \in \Gamma} A_i) &= \bigvee_{x \in A_i} N_x(\bigcup_{i \in \Gamma} A_i) \\ &= \bigvee_{x \in A_i} \bigvee_{x \in A_i} (N_x(A_i)) \\ &= (\bigvee_{x \in A_i} \mathcal{T}(A_i)) \\ &\ge \bigwedge_{x \in A_i} \mathcal{T}(A_i)\end{aligned}$$

and

$$\begin{aligned}(N_x^*)_{\mathcal{T}^*}(\bigcup_{i \in \Gamma} A_i) &= \bigwedge_{x \in A_i} \mathcal{T}^*(\bigcup_{i \in \Gamma} A_i) \\ &\le \bigwedge_{x \in A_i} \bigvee_{x \in A_i} (\mathcal{T}^*(A_i)) \\ &= \bigvee_{x \in A_i} \bigwedge_{x \in A_i} (\mathcal{T}^*(A_i)) \\ &= \bigvee_{x \in A_i} (N_x^*)_{\mathcal{T}^*}(A_i)\end{aligned}$$

(B)

$(DN-f_1)(N_x)_{\mathcal{T}}(A) = \mathcal{T}(A) \le \mathcal{T}^*(A) \to \perp = (N_x^*)_{\mathcal{T}^*}(A) \to \perp$,

$(N_x)_{\mathcal{T}}(X) = \mathcal{T}(X) = \top$, $(N_x)_{\mathcal{T}}(\phi) = \mathcal{T}(\phi) = \top$ and

$(N_x^*)_{\mathcal{T}^*}(X) = \mathcal{T}^*(X) = \perp$, $(N_x^*)_{\mathcal{T}^*}(\phi) = \mathcal{T}^*(\phi) = \perp \ \forall x \in X$,

(Boundary conditions)

$(DN-f_2)$ For each $A, B \in 2^X$ (Intersection property)

$$\begin{aligned}(N_x)_{\mathcal{T}}(A \cap B) &= \mathcal{T}(A \cap B) \\ &\ge \mathcal{T}(A) \wedge \mathcal{T}(B) \\ &= (N_x)_{\mathcal{T}}(A) \wedge (N_x)_{\mathcal{T}}(B) \\ &= N_x(A) \wedge N_x(B)\end{aligned}$$

and

$$\begin{aligned}(N_x^*)_{\mathcal{T}^*}(A \cap B) &= \mathcal{T}^*(A \cap B) \to \perp \\ &\le (\mathcal{T}^*(A)) \vee (\mathcal{T}^*(B)) \to \perp \\ &\ge (\mathcal{T}^*(A) \to \perp) \wedge (\mathcal{T}^*(B) \to \perp) \\ &\le (N_x^*)_{\mathcal{T}^*}(A) \vee (N_x^*)_{\mathcal{T}^*}(B)\end{aligned}$$

$(DN-u_3)$ whenever $x \notin A$, $(N_x^*)_{\mathcal{T}^*}(A) = \mathcal{T}^*(A) \to \perp = \top$ and $(N_x)_{\mathcal{T}}(A) = \perp$.

$(DN-u_4)$ Let $A = \cup_i A_i$

$$(N_x)_{\mathcal{T}}(A)=\mathcal{T}(\cup_i A_i)\le \bigwedge_{i\in I}(\mathcal{T}(A)$$
$$\le (\bigvee_{y\notin B}(\mathcal{T}(A)\vee \mathcal{T}(B)))$$
$$\le (\bigvee_{y\notin B}((N_y)_{\mathcal{T}}(A)\vee (N_x)_{\mathcal{T}}(B)$$

and

$$(N^*_x)_{\mathcal{T}*}(A)=\mathcal{T}*(\cup_i A_i)\to\perp$$
$$\le (\bigvee_i \mathcal{T}*(A_i))\to\perp$$
$$\ge (\bigwedge_{i\in I}(\mathcal{T}*(A_i)\to\perp)$$
$$=\bigwedge_{i\in I}(\bigwedge_{y\in B}(N^*_x)_{\mathcal{T}*}(A_i)\wedge N^*_x)_{\mathcal{T}*}(B)))$$

$(C)\mathcal{T}_{(N_x)_{\mathcal{T}}}(A)=\bigvee_{x\in A}(N_x)_{\mathcal{T}}(A)=\bigvee_{x\in A}\mathcal{T}(A)=\mathcal{T}(A)$ and

$\mathcal{T}^*_{(N^*_x)_{\mathcal{T}*}}(A)=\bigwedge_{x\in A}(N^*_x)_{\mathcal{T}*}(A)=\mathcal{T}*(A).$

Definition 4.2

Let X be a nonempy set. An element $(c,c^*)\in L^{X\times P(X)}$ is called an M-double fuzzy contiguity relation on X iff C fulfills the following axioms:

$(DC_1)C(x,A)\le c^*(x,A)\to\perp$, for every $x\in X$ and $A\in 2^X$.

$(Dc_2)c(x,A\cup B)\le c(x,A)\vee c(x,B)$, and

$c^*(x,A\cup B)\ge c^*(x,A)\wedge c^*(x,B)$ (Distributivity),

$(Dc_3)c(x,A)=\top$, and $c^*(x,A)=\perp$ whenever $x\in A$,

$(Dc_4)(\bigwedge_{y\in B}c(y,A))\wedge c(x,B)\le c(x,A)$ and

$\bigvee_{y\notin B}c^*(y,A))\vee c^*(x,B)\ge c^*(x,A)$ (Transitivity).

Theorem 4.2

Let $(X,\mathcal{T},\mathcal{T}^*)$ be an $(2,L)-$double fuzzifying topological space. We define the maps $(c_{\mathcal{T}},c^*_{\mathcal{T}*}):X\times 2^X\to L$ as follows $c_{\mathcal{T}}(x,A)=\mathcal{T}(X-A)\to\perp$, $c^*_{\mathcal{T}*}(x,A)=\mathcal{T}*(X-A)\to\perp$ Then the pair $(c_{\mathcal{T}},c^*_{\mathcal{T}*})$ is an M-double fuzzy contiguity relation on X induces by $(2,L)-$double fuzzifying topological space $(X,\mathcal{T},\mathcal{T}^*)$. Let (c,c^*) be an M-double fuzzy contiguity relation on X. Define $(\mathcal{T}_c,\mathcal{T}^*_{c*}):L^X\to L$ as follows: $\mathcal{T}_c(A)=c(x,X-A)\to\perp$, $\mathcal{T}^*_{c*}(A)=c^*(x,X-A)$. Then $(\mathcal{T}_c,\mathcal{T}^*_{c*})$ is an $(2,L)-$double fuzzifying topological space on X induces by an M-double fuzzy contiguity relation on X. Furthermore $\left(\mathcal{T}_{c_{\mathcal{T}}},\mathcal{T}^*_{c^*_{\mathcal{T}*}}\right)=(\mathcal{T},\mathcal{T}^*)$ and $(c_{\mathcal{T}_c},c^*_{\mathcal{T}^*_{c*}})=(c,c^*)$.

Proof

$(A)(DC1)\,For\ each\ A\in L^X, c^*_{\mathcal{T}^*}(x,A)\to\perp=$

$(\mathcal{T}^*(X-A)\to\perp)\to\perp\ge\mathcal{T}(X-A)\to\perp=C_{\mathcal{T}}(x,A),$

$(DC_2)\,For\ each\ A,B\in L^X,$

$$c_{\mathcal{T}}(x,A\cup B)=\mathcal{T}(X-(A-B))\to\perp$$
$$=\mathcal{T}((X-A)\cap(X-B))\to\perp$$
$$\le(\mathcal{T}(X-A)\to\perp)\vee\mathcal{T}(X-B)\to\perp)$$
$$=c_{\mathcal{T}}(x,A)\vee C_{\mathcal{T}}(x,B)$$

and

$$c^*_{\mathcal{T}^*}(x,A\cup B)=\mathcal{T}^*(X-(A\cup B))\to\perp$$
$$=\mathcal{T}^*((X-A)\cap(X-B))\to\perp$$
$$\le(\mathcal{T}^*(X-A))\vee\mathcal{T}^*(X-B)\to\perp)$$
$$\ge(\mathcal{T}^*(X-A)\to\perp)\vee\mathcal{T}^*(X-B)\to\perp)$$
$$\ge c^*_{\mathcal{T}^*}(x,A)\vee c^*_{\mathcal{T}^*}(x,B)$$

$(DC_3)\,For\ x\in A, c^*_{\mathcal{T}^*}(x,A)=\mathcal{T}^*(X-A)=\perp\ and\ c_{\mathcal{T}}(x,A)=\mathrm{T}$

(DC_4)

$$\mathcal{C}_{\mathcal{T}}(x,A)=\mathcal{T}(X-A)\to\perp$$
$$\ge\left(\bigvee_{y\notin B}(\mathcal{T}(X-A))\vee(\mathcal{T}(B))\right)\to\perp$$
$$\ge\left(\bigwedge_{y\in B}(\mathcal{T}(X-A)\to\perp)\right)\wedge(\mathcal{T}(X-B)\to\perp)$$

and

$$=\left(\bigwedge_{y\in B}\mathcal{C}_{\mathcal{T}}(y,A)\right)\wedge\mathcal{C}_{\mathcal{T}}(x,B),$$

$$c^*_{\mathcal{T}^*}(x,A)=\mathcal{T}^*(X-A)\to\perp$$
$$\le\left(\bigwedge_{y\notin B}\bigvee_{y\notin B}(\mathcal{T}^*(X-A))\wedge\mathcal{T}^*(X-B)\right)\to\perp$$
$$=\left(\bigvee_{y\notin B}c^*_{\mathcal{T}^*}(y,A)\right)\wedge c^*_{\mathcal{T}^*}(x,B)$$

(B)

$(DOI)\,for\ every\ x\in X\ and\ A\in 2^X, \mathcal{T}^*_{c^*}(A)\to\perp=$

$c^*(x,X-A)\to\perp\ge C(x,X-A)=\mathrm{T}\ge\mathcal{T}_C(A)$

$(DO2)\mathcal{T}_c(X)=c(X,\phi)\to\perp=\perp\to\perp=\mathrm{T},\mathcal{T}^*_{c^*}(\phi)=c^*(x,X)=\perp,\mathcal{T}^*_{c^*}(X)=\perp$

$(DO3)$

$$\mathcal{T}_{c*}(A\cap B)=c(x,X-(A\cap B))\to\perp$$
$$=c^*(x,(X-A)\cup(X-B))$$
$$\ge c^*(x,X-A))\wedge c^*(x$$
$$\le\mathcal{T}^*_{c*}(x,A)\wedge\mathcal{T}^*_{c*}(x,B)$$

and

$$\mathcal{T}_{c*}(A\cap B)=c(x,X-(A\cap B))\to\perp$$
$$=c^*(x,(X-A)\cup(X-B))$$
$$\ge c^*(x,X-A))\wedge c^*(x$$
$$\le\mathcal{T}^*_{c*}(x,A)\wedge\mathcal{T}^*_{c*}(x,B)$$

,

$(DO4)$

$$\mathcal{T}c(\bigcup_{i\in\Gamma}A_i)=C(x,X-(\bigcup_{i\in\Gamma}A_i))\to\perp$$
$$\le\left(\left(\bigwedge_{y\in B}C\left(y,\bigcap_{i\in\Gamma}(X-A_i)\right)\right)\wedge C(x,B)\right)\to\perp$$
$$\ge\left(\left(\bigvee_{y\in B}C\left(y,\bigcap_{i\in\Gamma}(X-A_i)\right)\right)\to\perp\right)\vee(C(x,B)\to\perp)$$
$$=\left(\bigwedge_{y\in B}\mathcal{T}_C(y,\bigcup_{i\in\Gamma}A_i)\right)\vee\mathcal{T}_C(x,X-B),$$
$$\ge\wedge_{i\in\Gamma}\mathcal{T}_C(A_i)$$

$$\mathcal{T}^*_{c^*}\left(\bigcup_{i\in\Gamma} A_i\right) = C^*\left(x, X - \left(\bigcup_{i\in\Gamma} A_i\right)\right)$$
$$\geq \left(\bigvee_{y\in B} C^*\left(y, X - \left(\bigcup_{i\in\Gamma} A_i\right)\right)\right) \vee C^*(x, B)$$

and

$$= \left(\bigvee_{y\in B} \mathcal{T}^*_{c^*}\left(x, \left(\bigcup_{i\in\Gamma} A_i\right)\right)\right) \vee \mathcal{T}^*_{c^*}(x, X - B),$$
$$\leq \bigvee_{i\in\Gamma} \mathcal{T}^*_{c^*}\left(A^i\right)$$

$(C)\mathcal{T}_{C_\mathcal{T}}(A) = C_\mathcal{T}(x, X - A) \to \perp = \mathcal{T}(A),\ \mathcal{T}^*_{c^*_{\mathcal{T}^*}}(A) = C^*_{\mathcal{T}^*}(x, X - A) = \mathcal{T}^*(A)$

And $C_{\mathcal{T}_C}(x, A) = \mathcal{T}_C(X - A) \to \perp = C(x, A)$ *and* $C^*_{\mathcal{T}^*_{c^*}}(x, A) = \mathcal{T}^*_{c^*}(X - A) \to \perp = C^*(x, A)$

Theorem 4.3

Let (X, N_x, N_x^*) be an $(2, L)$ double fuzzifying neighborhood system of x. We define the maps $\left(c_{(N_x)}, c^*_{(N_x^*)}\right): X \times 2^X \to L$ as follows:

$c_{(N_x)}(x, A) = (N_x)(X - A) \to \perp,\ c^*_{(N_x^*)}(x, A) = (N_x^*)(A) \to \perp$ Then the pair $\left(c_{(N_X)}, c^*_{(N_X^*)}\right)$ is anM-double fuzzy contiguity relation on X induces by $(2, L)$ double fuzzifying (X, N_X, N_x^*). Let (c, c^*)be an M-double fuzzy contiguity relation on X. Define $(N_X)_c, (N_x^*)_{c^*}: L^X \to L$ as follows: $(N_X)_c(A) = c(x, X - A) \to \perp,\ (N_x^*)_{c^*}(A) = c^*(x, X - A)$ Then $\left((N_X)_c, (N_x^*)_{c^*}\right)$ is an $(2, L)$-double fuzzifying neighborhood system induces by an M-double fuzzy contiguity relation on X. Furthermore $(N_X)_{c(N_X)}, (N_x^*)_{c^*_{N_X^*}} = (N_X, N_x^*)$ *and* $\left(c_{(N_X)_c}, c^*_{(N_X^*)_{c^*}}\right) = (c, c^*)$.

Proof (A)

(DC1) For each $A \in 2^X$, whenever

$x \in X, c_{(N_x)}(x, A) = (N_x)(X - A) = (N_x)(X - A) \to \perp \leq \left((N_x^*)_{C^*}(X - A) \to \perp\right) \to$

$\perp \leq \mathrm{T} \to \perp = \perp \leq \left((N_x^*)(A) \to \perp\right) \to \perp \leq c^*_{(N_x^*)}(X - A) \to \perp$. (DC2)

$$c_{(N_x)}(x, A \cup B) = (N_x)(X - (A \cup B)) \to \perp$$
$$= (N_x)((X - A) \cap (X - B)) \to \perp$$
$$\leq (N_x)(X - A) \to \perp) \vee (N_x)(X - B) \to \perp)$$
$$= c_{(N_x)}(x, A) \vee c_{(N_x)}(x, B)$$

and

$$c^*_{(N_x^*)}(x, A \cup B) = (N_x^*)((A \cup B)) \to \perp$$
$$\leq (N_x^*)((A) \vee (N_x^*)(B)) \to \perp$$
$$\geq (N_x^*)(A) \to \perp) \wedge (N_x^*)(B) \to \perp)$$
$$= (N_x^*)_{c^*}(x, A) \wedge (N_x^*)_{c^*}(x, B)$$

(DC3) whenever $x \in A$,

$c_{(N_x)}(x, A) = (N_x)(X - A) \to \perp = \mathrm{T}$, *and* $c^*_{(N_x^*)}(x, A) = \perp$ (DC4) $\forall B \in P(X)$,

$$c_{(N_x)}(x, A) = (N_x)(X - A) \to \perp$$
$$\leq \left(\bigvee_{y\notin B}\left((N_y)(X - A) \vee (N_x)(B)\right)\right) \to \perp$$
$$\geq \left(\bigwedge_{y\in B}\left((N_y)(X - A) \to \perp\right)\right) \vee \left((N_x)(X - B) \to \perp\right)$$
$$= \left(\bigwedge_{y\in B} c_{(N_y)}(y, A)\right) \wedge c_{(N_x)}(x, B),$$

and

$$c^*_{(N_x^*)}(x, A) = (N_x^*)(A) \to \perp$$
$$\geq \left(\bigwedge_{y\in B}(N_y^*)(A) \wedge \left((N_x^*)(B)\right)\right) \to \perp$$
$$\leq \left(\bigvee_{y\in B}\left((N_y^*)(A) \to \perp\right)\right) \vee \left(((N_x^*)(B) \to \perp\right)$$
$$= \left(\bigvee_{y\in B} c^*_{(N_y^*)}(y, A)\right) \vee c^*_{((N_x^*))}(x, B).$$
$$\geq \left(\bigvee_{y\in B} c^*_{(N_y^*)}(y, A)\right) \vee c^*_{((N_x^*))}(x, B).$$

(B)

$(DN - f1)$ forever $x \in X$ and

$(N_X)_c(N_X)(A) = c(x, X - A) \to \perp \leq c^*(x, X - A) \to \perp = (N_X^*)_{c^*}(A) \to$

$\perp, (N_X)c(X) = c(X, \phi) \to \perp = \perp \to \perp = \mathrm{T}, (N_X)_c(\phi) = \mathrm{T}$

$(N_X^*)_{c^*}(\phi) = c^*(x, X) = \perp, (N_X^*)_{c^*}(X) = \perp$

$(DN - f_2)$

$$(N_X)_c(A \cap B) = c(x, X - (A \cap B)) \to \perp$$
$$= c(x, (X - A) \cup (X - B)) \to \perp$$
$$\leq \left(c(x, (X - A))\right) \vee \left(c(x, (X - B))\right) \to \perp$$

and

$$\geq \left(c(x, (X - A)) \to \perp\right) \wedge \left(c(x, (X - B)) \to \perp\right)$$
$$= (N_X)_c(A) \wedge (N_X)_c(A)$$

$$(N_X^*)_{c^*}(A \cap B) = c^*(x, X - (A \cap B))$$
$$= c^*(x, (X - A) \cup (X - B))$$
$$\geq \left(c^*(x, (X - A))\right) \vee \left(c^*(x, (X - B))\right)$$
$$= (N_X^*)_{c^*}(A) \wedge (N_X^*)_{c^*}(B)$$
$$\leq (N_X^*)_{c^*}(A) \vee (N_X^*)_{c^*}(B)$$

$(DN - u_3)$ whenever $x \notin A$,

$(N_X)_c(A) = c(x, X - A) \to \perp = \mathrm{T} \to \perp = \perp$, *and* $(N_X^*)_{c^*}$

$(DN - u_4)$ For each $x \in A$, and $A, B \in 2^X$

$$(N_X)_c(A) = c(x, X - A) \to \perp$$
$$\geq \left(\bigwedge_{y\in B}\left(c(y, X - A) \wedge c(x, X - B)\right)\right) \to \perp$$
$$\leq \left(\bigvee_{y\notin B}\left(c(y, X - A) \to \perp\right)\right) \vee \left(c(x, X - B) \to \perp\right)$$
$$= \left(\bigvee_{y\notin B}(N_y)_c(A)\right) \vee (N_X)_c(B),$$

and

$$(N_X^*)_{c^*}(A) = c^*(x, X - A)$$
$$\leq \left(\bigvee_{y\notin B}\left(c^*(y, A) \vee c^*(x, B)\right)\right)$$

(A)

$$= \left(\bigvee_{y\notin B}\left((N_X^*)_{c^*}(X - A)\right) \vee \left((N_X^*)_{c^*}(X - B)\right)\right)$$
$$\geq \left(\bigwedge_{y\in B}(N_y^*)_{c^*}(A)\right) \vee (N_X^*)_{c^*}(B)$$

$(C)(N_X)_{c_{(N_X)}}(A)=c_{(N_X)}(x,X-A)\to\bot=(N_X)(A),(N^*_x)_{c^*_{(N^*_X)}}(A)=(N^*_x)(A)$

$and\ c_{(N_X)c}(A)=(N_X)_c(X-A)\to\bot=(c(x,A)\to\bot)\to\bot=c(x,A),c^*_{(N^*_x)_c}(A)=c^*(x-A).$

Definition 3.3. Let X be a nonempty set. A map $((\ )^\circ,(\ )^{*\circ}):2^X\to L^X$ is called an $(2,L)$-double fuzzifying interior operator if $((\ )^\circ,(\ )^{*\circ})$ satisfies the following conditions:

$(1^\circ)(A)^\circ=(A)^{*\circ}\to\bot\ and\ (X)^\circ=\mathrm{T},(\phi)^{*\circ}=\bot$

$(2^\circ)(A\cap B)^\circ=(A)^\circ\wedge(B)^\circ,(A\cap B)^{*\circ}=(A)^{*\circ}\vee(B)^{*\circ},$

$(3^\circ)(A)^\circ\le A,A\le(A)^{*\circ},$

$(4^\circ)(A)^\circ(x)\le\vee_{y\notin B}((A)^\circ(y)\vee(B)^\circ(x)),(A)^{*\circ}(x)\le\wedge_{y\notin B}((A)^{*\circ}(y)\wedge(B)^{*\circ}(x)).$

Theorem 4.4

Let (N_x,N^*_x) be an (2,M)-double fuzzifying neighborhood system of x. We Define $((\ )^\circ,(\ )^{*\circ})_{(N_X,N^*_X)}:2^X\to L^X$ as follows: $(N_X)_{(\ )^\circ}(A)=(A)^\circ(x),(N^*_X)_{(\ )^{*\circ}}(A)=(A)^{*\circ}(x)$. Then $((\ )^\circ,(\ )^{*\circ})_{(N_X,N^*_X)}$ is an M-double fuzzifying interior operator induces by (2,M)-double fuzzifying neighborhood system of x. Let $((\ )^\circ,(\ )^{*\circ})$ be an M-double fuzzifying interior operator.

We Define $(N_X,N^*_X)_{((\ )^\circ,(\ )^{*\circ})}:2^X\to L^X$ as:

$(\ )^\circ_{N_X}(A)=N_XA,(\ )^{*\circ}_{N^*_X}(A)=N^*_X(A).$

Then $(N_X,N^*_X)_{((\ )^\circ,(\ )^{*\circ})}$ is an (2,M)-double fuzzifying neighborhood systemof x induces by M-double fuzzifying interior operator $((\ )^\circ,(\ )^{*\circ})$ on X. Moreover $(N_X,N^*_X)_{((\ )^\circ,(\ )^{*\circ})_{(N_X,N^*_X)}}=(N_X,N^*_X)$ and $((\ )^\circ,(\ )^{*\circ})(N_X,N^*_X)_{((\ )^\circ,(\ )^{*\circ})}=((\ )^\circ,(\ )^{*\circ}).$

Proof

(A) For each $A\in2^X$,

$(DN-f_1)(N_X)_{(\ )^\circ}(A)=(A)^\circ(x)=(A)^{*\circ}\to\bot\le(N^*_X)_{(\ )^{*\circ}}(A)\to\bot.$

$And,(N_X)_{(\ )^\circ}(X)=(X)^\circ(x)=\mathrm{T},(N_X)_{(\ )^\circ}(\phi)=(\phi)^\circ(x)\le(\phi)^{*\circ}\to\bot=\bot\to\bot=\mathrm{T}$

$and\ (N^*_X)_{(\ )^\circ}(X)=(X)^{*\circ}(x)=(X)^\circ\to\bot=\mathrm{T}\to\bot=\bot,(N^*_X)_{(\ )^\circ}(\phi)=(\phi)^{*\circ}(x)=\bot,$

$(DN-f_2)$ for each $A,B\in2^X$

$(N_X)_{(\ )^\circ}(A\cap B)=(A\cap B)^\circ\ge(A)^\circ(x)\wedge(B)^\circ=(N_X)_{(\ )^\circ}(A)\wedge(N_X)_{(\ )^\circ}(B)$

$and\ (N^*_X)_{(\ )^{*\circ}}(A\cap B)=(A\cap B)^{*\circ}\le(A)^{*\circ}(x)\vee(B)^{*\circ}=(N^*_X)_{(\ )^{*\circ}}(A)\vee(N^*_X)_{(\ )^{*\circ}}(B)$

$(DN-u_3)$ whenever

$x\notin A(N_X)_{(\ )^{*\circ}}(A)=(A)^\circ=\bot\ and\ (N^*_X)_{(\ )^{*\circ}}(A)^{*\circ}=\mathrm{T}$ $(DN-u_4)$ for each $x\in A,$

$(N_X)_{(\ )^{*\circ}}(A)=(A)^\circ\le\vee_{y\notin B}((A)^\circ(y)\vee(B)^\circ(x))=\vee_{y\notin B}((N_y)_{(\ )^{*\circ}}(A)(y)\vee(N_X)_{(\ )^\circ}(B)(x))$

$And\ (N^*_X)_{(\ )^{*\circ}}(A)=(A)^{*\circ}\le\wedge_{y\notin B}((N^*_y)_{(\ )^{*\circ}}(A)(y)\wedge(N^*_X)_{(\ )^{*\circ}}(B)(x))$

$((B)(1^\circ)\ for\ each\ A\in2^X\ (\ )^\circ_{N_X}(A)=N_X(A)\le N^*_X(A)\to\bot=(A)^{*\circ}(x)\to\bot=(\ )^{*\circ}_{N^*_X}(A)\to\bot$

$and\ \forall x\in X,(\ )^\circ_{N_X}(X)=N_X(X)=\mathrm{T},(\ )^\circ_{N_X}(\phi)=N_X(\phi)=\mathrm{T}$

$and\ (\ )^{*\circ}_{N^*_X}(X)=N^*_X(X)=\bot,(\ )^{*\circ}_{N^*_X}(\phi)=N^*_X(\phi)=\bot$

$(2^\circ)\ for\ each\ A,B\in2^X,(\ )^\circ_{N_X}(A\cap B)=N_X(A\cap B)\ge N_X(B)=(\ )^\circ_{N_X}(A)\wedge(\ )^\circ_{N_X}(B).$

$and\ (\ )^{*\circ}_{N^*_X}(A\cap B)=N^*_X(A\cap B)\le N^*_X(A)\vee N^*_X(B)=(\ )^{*\circ}_{N_X}(A)\vee(\ )^{*\circ}_{N_X}(B).$

$(3^\circ)(\ )^\circ_{N_x}(A)=N_x(A)\le A(x)\ and\ A(x)\le N^*_x(A)=(\ )^{*\circ}_{N^*_x}(A)\ whenever\ x\notin A.$

$(4^\circ)\ For\ each\ x\in A,(\ )^\circ_{N_x}(A)=N_x(A)\le\vee_{y\notin B}(N_y(A)\vee N_x(B))=\vee_{y\notin B}((\ )^\circ_{N_y}(A)(y)\vee(\ )^\circ_{N_x}(B)(x))$

$and\ (\ )^{*\circ}_{N^*_x}(A)=N^*_x(A)\ge\wedge_{y\notin B}(N^*_y(A)\wedge N^*_x(B))=\wedge_{y\notin B}((\ )^{*\circ}_{N^*_y}(A)\wedge(\ )^{*\circ}_{N^*_x}(B)).\forall B\in P(X).$

$(C)(N_X,N^*_X)_{((\ )^\circ,(\ )^{*\circ})_{(N_X,N^*_X)}}(A)=((\ )^\circ,(\ )^{*\circ})_{(N_X,N^*_X)}(A)=(N_x,N^*_x)(A),and$

$((\ )^\circ,(\ )^{*\circ})_{(N,N^*)((\ )^\circ,(\ )^{*\circ})}(A)=(N_X,N^*_X)_{((\ )^\circ,(\ )^{*\circ})}(A)=((\ )^\circ,(\ )^{*\circ})(A).$

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