

# The d'Alembertian operator and Maxwell's equations

Claude Michael Cassano

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**ABSTRACT**

The d'Alembertian is a linear second order differential operator, typically in four independent variables. The time-independent version (in three independent (space) variables) is called the Laplacian operator. When its action on a function or vector vanishes, the resulting equation is called the wave equation (or Laplace's equation). When it's action is identified with a non-zero function (or vector function) the resulting equation is called

Poisson's equation. This equation is fundamental to and of great importance in field theory. The focus of this tutorial is to demonstrate that when acting on a four-vector, the d'Alembertian may be factored into two 4 x 4 differential matrices. This d'Alembertian operator factorization of a four-vector into two 4 x 4 differential matrices is not merely another form of expressing Maxwell's equations; but remarkably, yields a quantum and unified field theory generalization.

**Key Words:** Maxwell's equations; Unified field theory; d'Alembertian; Differential operator; Mathematical physics section; Isoperimetric theorem; Curvature

**INTRODUCTION**

The d'Alembertian is a linear second order differential operator, typically in four independent variables. The time-independent version (in three independent (space) variables) is called the Laplacian operator. When it's action on a function or vector vanishes, the resulting equation is called the wave equation (or Laplace's equation). When it's action is identified with a non-zero function (or vector function) the resulting equation is called Poisson's equation. This equation is fundamental to and of great importance in field theory. The focus of this tutorial is to demonstrate that when acting on a four-vector, the d'Alembertian may be factored into two 4 x 4 differential matrices.

Many field theory books denote electromagnetic field tensor and dual tensor (including [1-3]), respectively, in 4 x 4 matrix form, as follows:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E^1 & -E^2 & -E^3 \\ E^1 & 0 & -B^3 & B^2 \\ E^2 & B^3 & 0 & -B^1 \\ E^3 & -B^2 & B^1 & 0 \end{pmatrix}, \quad \tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3 & -E^2 \\ B^2 & -E^3 & 0 & E^1 \\ B^3 & E^2 & -E^1 & 0 \end{pmatrix}$$

yielding Maxwell's equation as:  $\partial_\mu F^{\mu\nu} = j^\nu, \partial_\mu \tilde{F}^{\mu\nu} = 0$

Noting the sum and difference of these matrices, then defining: (Using the same Einstein summation convention)

$$\partial_\nu^\mu \equiv \begin{cases} 1, \mu = \nu \\ 0, \text{otherwise} \end{cases}, \text{ (the Kroenecker delta function)}$$

$$\square \equiv \partial_\mu \partial_\mu$$

$$\square^* \equiv (-1) \delta_0^\mu \partial_\mu \partial_\mu$$

$$f \equiv u_\mu f^\mu$$

$$f^* \equiv (-1) \delta_0^\mu u_\mu f^\mu$$

$$\nabla \equiv u_\mu \partial_\mu$$

$$\nabla^* \equiv (-1) \delta_0^\mu u_\mu \partial_\mu$$

$$\nabla \cdot f = \partial_\mu f^\mu$$

$$\nabla \cdot f^* = (-1) \delta_0^\mu u_\mu \partial_\mu f^\mu = \nabla^* \cdot f$$

$$E = u_\mu (-\partial_0 f^\mu - \partial_\mu f^0), \quad (\mu \in N, 1 \leq \mu \leq 3), \quad (1)$$

and:

$$B = u_\mu (-1)^{\mu+1} [\partial_{2-T_0(\mu)} f^{3-T_0(\mu-1)} - \partial_{3-T_0(\mu-1)} f^{2-T_0(\mu)}], \quad (\mu \in N, 1 \leq \mu \leq 3), \quad (2)$$

$$T_0(j) \equiv \frac{1}{2} (j-1 + \delta_{(-1)^j}^1), \quad (j \in N)$$

$B - E :$

$$\begin{pmatrix} \partial_0 & -\partial_3 & \partial_2 & \partial_1 \\ \partial_3 & \partial_0 & -\partial_1 & \partial_2 \\ -\partial_2 & \partial_1 & \partial_0 & \partial_3 \\ \partial_1 & \partial_2 & \partial_3 & -\partial_0 \end{pmatrix} \begin{pmatrix} f^1 \\ f^2 \\ f^3 \\ f^0 \end{pmatrix} = \begin{pmatrix} B^1 - E^1 \\ B^2 - E^2 \\ B^3 - E^3 \\ \nabla \cdot f^* \end{pmatrix} \quad (3)$$

and:

$$J \equiv \begin{pmatrix} J^1 \\ J^2 \\ J^3 \\ J^0 \end{pmatrix} \equiv \begin{pmatrix} \partial_0 & \partial_3 & -\partial_2 & \partial_1 \\ -\partial_3 & \partial_0 & \partial_1 & \partial_2 \\ \partial_2 & -\partial_1 & \partial_0 & \partial_3 \\ \partial_1 & \partial_2 & \partial_3 & -\partial_0 \end{pmatrix} \begin{pmatrix} \partial_0 & -\partial_3 & \partial_2 & \partial_1 \\ \partial_3 & \partial_0 & -\partial_1 & \partial_2 \\ -\partial_2 & \partial_1 & \partial_0 & \partial_3 \\ \partial_1 & \partial_2 & \partial_3 & -\partial_0 \end{pmatrix} \begin{pmatrix} f^1 \\ f^2 \\ f^3 \\ f^0 \end{pmatrix} = \begin{pmatrix} \square f^1 \\ \square f^2 \\ \square f^3 \\ \square f^0 \end{pmatrix} \quad (4)$$

Warburton AVE 4, Santa Clara, CA 95051, United States.

Correspondence: Claude Michael Cassano, Warburton AVE 4, Santa Clara, CA 95051, United States, Tel: 6506031264; E-mail: cloudmichael@hotmail.com

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$B + E :$

$$\begin{pmatrix} -\partial_0 & -\partial_3 & \partial_2 & -\partial_1 \\ \partial_3 & -\partial_0 & -\partial_1 & -\partial_2 \\ -\partial_2 & \partial_1 & -\partial_0 & -\partial_3 \\ -\partial_1 & -\partial_2 & -\partial_3 & \partial_0 \end{pmatrix} \begin{pmatrix} f^1 \\ f^2 \\ f^3 \\ f^0 \end{pmatrix} = \begin{pmatrix} B^1 + E^1 \\ B^2 + E^2 \\ B^3 + E^3 \\ -\nabla \cdot f^* \end{pmatrix} \quad (5)$$

and:

$$J \equiv \begin{pmatrix} J^1 \\ J^2 \\ J^3 \\ J^0 \end{pmatrix} \equiv \begin{pmatrix} -\partial_0 & \partial_3 & -\partial_2 & -\partial_1 \\ -\partial_3 & -\partial_0 & \partial_1 & -\partial_2 \\ \partial_2 & -\partial_1 & -\partial_0 & -\partial_3 \\ -\partial_1 & -\partial_2 & -\partial_3 & \partial_0 \end{pmatrix} \begin{pmatrix} f^1 \\ f^2 \\ f^3 \\ f^0 \end{pmatrix} = \begin{pmatrix} \square f^1 \\ \square f^2 \\ \square f^3 \\ \square f^0 \end{pmatrix} \quad (6)$$

Then,

$$J^1 \equiv \partial_0(B^1 - E^1) + \partial_3(B^2 - E^2) - \partial_2(B^3 - E^3) + \partial_1(\nabla \cdot f^*)$$

$$J^2 \equiv -\partial_3(B^1 - E^1) + \partial_0(B^2 - E^2) + \partial_1(B^3 - E^3) + \partial_2(\nabla \cdot f^*)$$

$$J^3 \equiv \partial_2(B^1 - E^1) - \partial_1(B^2 - E^2) + \partial_0(B^3 - E^3) + \partial_3(\nabla \cdot f^*)$$

$$J^0 \equiv \partial_1(B^1 - E^1) + \partial_2(B^2 - E^2) + \partial_3(B^3 - E^3) - \partial_0(\nabla \cdot f^*)$$

and:

$$J^1 \equiv -\partial_0(B^1 + E^1) + \partial_3(B^2 + E^2) - \partial_2(B^3 + E^3) + \partial_1(\nabla^* \cdot f)$$

$$J^2 \equiv -\partial_3(B^1 + E^1) - \partial_0(B^2 + E^2) + \partial_1(B^3 + E^3) + \partial_2(\nabla^* \cdot f)$$

$$J^3 \equiv \partial_2(B^1 + E^1) - \partial_1(B^2 + E^2) - \partial_0(B^3 + E^3) + \partial_3(\nabla^* \cdot f)$$

$$J^0 \equiv -\partial_1(B^1 + E^1) - \partial_2(B^2 + E^2) - \partial_3(B^3 + E^3) - \partial_0(\nabla^* \cdot f)$$

(remembering that  $\nabla^* \cdot f = \nabla \cdot f^*$ )

Subtracting:

$$0 \equiv \partial_0(2B^1) + \partial_3(-2E^2) - \partial_2(-2E^3)$$

$$0 \equiv -\partial_3(-2E^1) + \partial_0(2B^2) + \partial_1(-2E^3)$$

$$0 \equiv -\partial_1(2B^1) - \partial_2(2B^2) - \partial_3(2B^3)$$

$$0 \equiv -\partial_1(2B^1) - \partial_2(2B^2) - \partial_3(2B^3)$$

These may be expressed as:

$$\nabla \times E + \partial_0 B = 0 \text{ and } \nabla \cdot B = 0 \quad (7)$$

are the Homogeneous Maxwell's Equations,

Satisfied identically, due to the definitions of  $\mathbf{E}$  and  $\mathbf{B}$ .

Adding:

$$2J^2 \equiv -\partial_3(2B^1) + \partial_0(-2E^2) + \partial_1(2B^3) + 2\partial_2(\nabla \cdot f^*)$$

$$2J^2 \equiv -\partial_3(2B^1) + \partial_0(-2E^2) + \partial_1(2B^3) + 2\partial_2(\nabla \cdot f^*)$$

$$2J^3 \equiv \partial_2(2B^1) - \partial_1(2B^2) + \partial_0(-2E^3) + 2\partial_3(\nabla \cdot f^*)$$

$$2J^0 \equiv \partial_1(-2E^1) + \partial_2(-2E^2) + \partial_3(-2E^3) - 2\partial_0(\nabla \cdot f^*)$$

Continuing, from (6):

$$J \equiv \begin{pmatrix} J^1 \\ J^2 \\ J^3 \\ J^0 \end{pmatrix} \equiv \begin{pmatrix} -\partial_0 & \partial_3 & -\partial_2 & -\partial_1 \\ -\partial_3 & -\partial_0 & \partial_1 & -\partial_2 \\ \partial_2 & -\partial_1 & -\partial_0 & -\partial_3 \\ -\partial_1 & -\partial_2 & -\partial_3 & \partial_0 \end{pmatrix} \begin{pmatrix} f^1 \\ f^2 \\ f^3 \\ f^0 \end{pmatrix} = \begin{pmatrix} \square f^1 \\ \square f^2 \\ \square f^3 \\ \square f^0 \end{pmatrix} \quad (7)$$

$$= \begin{pmatrix} -\partial_0 & \partial_3 & -\partial_2 & -\partial_1 \\ -\partial_3 & -\partial_0 & \partial_1 & -\partial_2 \\ \partial_2 & -\partial_1 & -\partial_0 & -\partial_3 \\ -\partial_1 & -\partial_2 & -\partial_3 & \partial_0 \end{pmatrix} \begin{pmatrix} B^1 + E^1 \\ B^2 + E^2 \\ B^3 + E^3 \\ -\nabla^* \cdot f \end{pmatrix} \quad (8)$$

$$= \begin{pmatrix} -\partial_0(B^1 + E^1) + \partial_3(B^2 + E^2) - \partial_2(B^3 + E^3) - \partial_1(-\nabla^* \cdot f) \\ -\partial_3(B^1 + E^1) - \partial_0(B^2 + E^2) + \partial_1(B^3 + E^3) - \partial_2(-\nabla^* \cdot f) \\ \partial_2(B^1 + E^1) - \partial_1(B^2 + E^2) - \partial_0(B^3 + E^3) - \partial_3(-\nabla^* \cdot f) \\ -\partial_1(B^1 + E^1) - \partial_2(B^2 + E^2) - \partial_3(B^3 + E^3) + \partial_0(-\nabla^* \cdot f) \end{pmatrix} \quad (9)$$

$$= - \begin{pmatrix} (\partial_0 B^1 + \partial_2 E^3 - \partial_3 E^2) + (\partial_0 E^1 + \partial_2 B^3 - \partial_3 B^2) - \partial_1(\nabla^* \cdot f) \\ (\partial_0 B^2 - \partial_1 E^3 + \partial_3 E^1) + (\partial_0 E^2 - \partial_1 B^3 + \partial_3 B^1) - \partial_2(\nabla^* \cdot f) \\ (\partial_0 B^3 + \partial_1 E^2 - \partial_2 E^1) + (\partial_0 E^3 + \partial_1 B^2 - \partial_2 B^1) - \partial_3(\nabla^* \cdot f) \\ (\partial_1 B^1 + \partial_2 B^2 + \partial_3 B^3) + (\partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3) + \partial_0(\nabla^* \cdot f) \end{pmatrix} \quad (10)$$

$$= - \begin{pmatrix} (\partial_0 E^1 + \partial_2 B^3 - \partial_3 B^2) - \partial_1(\nabla^* \cdot f) \\ (\partial_0 E^2 - \partial_1 B^3 + \partial_3 B^1) - \partial_2(\nabla^* \cdot f) \\ (\partial_0 E^3 + \partial_1 B^2 - \partial_2 B^1) - \partial_3(\nabla^* \cdot f) \\ (\partial_1 E^1 + \partial_2 E^2 + \partial_3 E^3) + \partial_0(\nabla^* \cdot f) \end{pmatrix} \quad (11)$$

At this point, one may define:

$$x^0 = ict, \quad f^0 = i\phi, \quad D = iE$$

Then:

$$j^1 \equiv -J^1 = \frac{1}{c} \partial_t D^1 + \partial_2 B^3 - \partial_3 B^2 - \partial_1(\nabla \cdot f^*)$$

$$j^0 \equiv iJ^0 = \partial_1 D^1 + \partial_2 D^2 + \partial_3 D^3 - \frac{1}{c} \partial_t(\nabla \cdot f^*) \quad (12)$$

$$j^3 \equiv -J^3 = \frac{1}{c} \partial_t D^3 + \partial_1 B^2 - \partial_2 B^1 - \partial_3(\nabla \cdot f^*)$$

$$j^0 \equiv iJ^0 = \partial_1 D^1 + \partial_2 D^2 + \partial_3 D^3 - \frac{1}{c} \partial_t(\nabla \cdot f^*) \quad (12)$$

These are the Inhomogeneous Maxwell's Equations

Notable definitions for current/charge densities are (for the various, respective guages):

Representation I:

$$J \equiv \begin{pmatrix} J^1 \\ J^2 \\ J^3 \\ J^0 \end{pmatrix} = \begin{pmatrix} -\square f^1 + \partial_1(\nabla \cdot f^*) \\ -\square f^2 + \partial_2(\nabla \cdot f^*) \\ -\square f^3 + \partial_3(\nabla \cdot f^*) \\ -\square f^0 - \partial_0(\nabla \cdot f^*) \end{pmatrix} = \begin{pmatrix} \partial_0 E^1 + \partial_2 B^3 - \partial_3 B^2 \\ \partial_0 E^2 - \partial_1 B^3 + \partial_3 B^1 \\ \partial_0 E^3 + \partial_1 B^2 - \partial_2 B^1 \\ \partial_1 E^1 + \partial_2 E^2 + \partial_3 B^3 \end{pmatrix} \quad (13)$$

Representation II:

$$J \equiv \begin{pmatrix} J^1 \\ J^2 \\ J^3 \\ J^0 \end{pmatrix} = \begin{pmatrix} -\square *f^1 + \partial_1(\nabla \cdot f) \\ -\square *f^2 + \partial_2(\nabla \cdot f) \\ -\square *f^3 + \partial_3(\nabla \cdot f) \\ -\square *f^0 + \partial_0(\nabla \cdot f) \end{pmatrix} = \begin{pmatrix} -\partial_0 E^1 + \partial_2 B^3 - \partial_3 B^2 \\ -\partial_0 E^2 - \partial_1 B^3 + \partial_3 B^1 \\ -\partial_0 E^3 + \partial_1 B^2 - \partial_2 B^1 \\ \partial_1 E^1 + \partial_2 E^2 + \partial_3 B^3 \end{pmatrix} \quad (14)$$

Note: The field equations are invariant for all  $f$  such that:

In representation I:

$$\nabla \cdot f^* = \partial_1 f^1 + \partial_2 f^2 + \partial_3 f^3 - \partial_0 f^0 = constant \quad (15)$$

In representation II (Lorentz representation):

$$\nabla \cdot f = \partial_1 f^1 + \partial_2 f^2 + \partial_3 f^3 + \partial_0 f^0 = constant \quad (16)$$

(This is commonly called the Lorentz Gauge)

Clearly, another representation comes to mind:

$$\frac{1}{2}[(\nabla \cdot f) + (\nabla \cdot f^*)] = \partial_1 f^1 + \partial_2 f^2 + \partial_3 f^3 = constant \quad (17)$$

(Commonly called the Coulomb or Radiation Gauge)

This d'Alembertian operator factorization of a four-vector into two  $4 \times 4$  differential matrices is not merely another form of expressing Maxwell's equations; but remarkably, yields a quantum and unified field theory generalization.

#### REFERENCES

1. Ryder LH. Quantum field theory. Cambridge University Press. Cambridge, UK. 1986.
2. Kovetz A. Electromagnetic theory. Oxford University Press, Inc, New York. 2000.
3. Graham W. The Cambridge handbook of physics formulas. Cambridge University Press, New York. 2000.