Translational shape-invariance of radial Jacobi-reference potential under two sequential Darboux transformations

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ABSTRACT

The paper represents a further development of the mathematical formalism relating the conventional quantum-mechanical theory of the exactly-solvable translationally shape-invariant (TSI) potentials to the preservation of the form of the corresponding Rational Canonical Sturm-Liouville Equation (RCSLE) under the Darboux deformation of its Liouville potential using the so-called 'basic' solution as the Transformation Function (TF). Since these 'Liouville-Darboux transformations' (as we term them) simply shift one of the parameters by 1 we refer to this feature of the mentioned RCSLE as its 'translational form-invariance'.

The main purpose of this paper is to demonstrate that, by analogy with its TSI limit represented by the hyperbolic Pöschl-Teller potential, the radial potential exactly solvable via hypergeometric functions (termed 'radial Jacobi-reference potential in the paper) has two pairs of quasi-rational solutions such that their analytical continuations do not have zeros at any regular point in the complex plane. It is essential that the Characteristic Exponents (ChExps) of these four 'basic' solutions for the pole at the origin differ only by sign and can be then grouped into the pairs via the requirement that the paired solutions share the same characteristic exponent for the mentioned singularity. Each pair of the basic solutions is then used as seed functions for the second-order Darboux-Crum Transformation (DCT) of the radial potential in question. It is proven that both transformations simply shift by 2 the exponent difference for the pole of the RCSLE at the origin while keeping two other parameters unchanged. In other words, the DCT in question brings us back to the initial radial potential by either deleting the ground-energy state or inserting a new one.

The important novel element of our approach is the conversion of the Crum Wronskian (CW) to the Krein Determinant (KD) which makes it possible to express the eigenfunctions of the transformed RCSLE in terms of polynomial solutions of a Heine-type differential equation with degree-dependent exponential parameters. As an illustration of the Darboux-Crum-Krein theory of CSLEs put forward by the author we make use of the Krein representation to explicitly confirm that the state-deleting DCT of the Jost solution simply shifts the third parameter of the corresponding hypergeometric series by 2 as the direct consequence of the translational form-invariance of the RCSLE under the DCT in question.

Key Words: *Canonical Sturm-Liouville equation; Liouville transformation; translationally shape-invariant potential; Darboux-Crum-Krein theory; Jost solution*

INTRODUCTION

Half a century ago the author (Natanzon GA. 1978-79) performed a detailed analysis of the radial ('implicit') potential exactly solvable via a superposition of two hypergeometric series and in particular presented the analytical formulas for both Jost function and S-matrix element for the s-wave scattering. The potential was re-discovered by Ginocchio (1985) in a slightly different context a decade later. Namely he constructed a solvable radial potential with a varying mass which as immediately pointed to by Wu (1985; Wu Je et al 1989) turned into our potential in the constant-mass limit. In Appendix A we present the direct proof that the formula obtained by Ginocchio for the S-matrix element precisely agrees with the one derived in [1-5].

The next important development was made by (Lévai et al 1997) who applied double-step Darboux Transformations (DTs) to Ginocchio's potential re-written in a modified form (see Appendix B below for details). They took advantage of the analytical formulas for the scattering amplitudes of the s-wave scattering in the 'modified' Ginocchio potential and its double-step Darboux transforms (DTs) to demonstrate the phase-equivalence of the two potentials for certain combinations of the two seed functions used for the given Darboux-Crum transformation [6,7].

The main purpose of this publication is to show that the exactly-solvable radial potential in question (while not being 'shape-invariant' in Gendenshtein's terms (1983) preserves its form under specially

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chosen sequential DTs. To do it we first need to shift the focus back to the canonical Sturm-Liouville Equation (CSLE) used to generate the potential of our interest via the Liouville transformation [8,9].

We (Natanson G; 2016) term this CSLE as 'Jacobi-reference' ($\mathfrak{I}$Ref) since its eigenfunctions are expressible in terms of classical Jacobi polynomials with degree-dependent indexes assuming that the corresponding spectral problem is formulated on the finite interval [10,11, 1, 2]. The cornerstone of our approach is the characterization of the meromorphic density function

$$ {}_1\rho[z;T_K] = \frac{T_K[z]}{4z^2(1-z)^2} \text{ (K=0, 1, or 2)} \qquad (1) $$

by the orders of zeros of its polynomial numerator of the degree K at the poles of the $\mathfrak{I}$Ref CSLE in the finite plane. As discussed in detail (Natanson G; 2021) in the CSLE is 'translationally form-invariant' (TFI) iff the mentioned 'tangent' polynomial (TP) $T_K[z]$ has zeros only at the CSLE poles 0 and 1 [12]. This is the necessary and sufficient condition for the corresponding Liouville potential to become 'shape-invariant' in the conventional sense [8].

The concept of the 'translational form-invariance' put forward (Natanson G; 2021) in is essentially based on the existence of the 'basic' solutions

$$ {}_1\phi_{\dagger,0}[z] := z^{\frac{1}{2}\lambda_{0;\dagger,0}+\frac{1}{2}}(1-z)^{\frac{1}{2}\lambda_{1;\dagger,0}+\frac{1}{2}} \qquad (2) $$

such that their analytical continuations into the complex plane remain finite at any regular point of the $\mathfrak{I}$Ref CSLE. We say that the CSLE is TFI if it preserves its form under the so-called (Schnizer WA, et al 1993-1994) generalized Darboux' transformations such that the (Rudyak BV 1986) function,

$$ {}^{*}_{1}\phi_{\dagger,0}[z] := {}_1\rho^{-1}[z;T_K] / {}_1\phi_{\dagger,0}[z] \qquad (3) $$

is a solution of the transformed CSLE. It can be shown that these transformations of the CSLE are equivalent to the following three-step operation:

1. the Liouville transformation of the generic CSLE to the Schrödinger equation via the change of variable z(x) satisfying the first-order Ordinary Differential Equation (ODE):

   $$ z'(x;T_K) = {}_1\rho^{-\frac{1}{2}}[z(x;T_K);T_K] \qquad (4) $$

   where prime denotes the derivative with respect to x;
2. the Darboux deformation of the resultant Liouville potential with the Transformation Function (TF):

   $$ \psi_{\dagger,0}(x;T_K) := {}_1\rho^{-\frac{1}{2}}[z(x;T_K);T_K]\,{}_1\phi_{\dagger,0}[z(x;T_K)]; $$
   (5)
3. the reverse Liouville transformation of the Schrödinger equation with the deformed potential to the new CSLE with the unchanged density function ${}_1\rho[z;T_K]$.

We (Natanson G, 2016) suggested to refer to this operation as "Liouville-Darboux transformation" (LDT), keeping in mind that various authors give the term 'generalized Darboux transformation' completely different meanings [16].

Let i_r be the order of the TP zero at the singular point r = 0 or 1. Then the i_0,i_1;K- $\mathfrak{I}$Ref CSLE is TFI iff

$i_0 + i_1 = K$. (6)

Assuming (Natanzon GA, 1971) that all the Liouville transformations are performed on the finite interval (0, 1), there are the four TFI CSLEs associated with the four TSI potentials. Namely the Rosen-Morse (RM) potential (Rosen N, 1932) (the only TSI potential on the line) can be obtained by applying the Liouville transformation to the 0,0;0- $\mathfrak{I}$Ref CSLE [17].

There are two radial TSI potentials: the Manning-Rosen (MR) and hyperbolic Poschl-Teller (*h*-PT) potentials (Manning MF et al; Poschl G et al ;1933) which can be generated by applying the Liouville transformations to the 2,0;2- $\mathfrak{I}$Ref and 1,0;1- $\mathfrak{I}$Ref CSLEs accordingly [18,19].

And finally, the trigonometric Poschl-Teller (*t*-PT) potential (Poschl G et al ;1933) (originally discussed by Darboux et along before the birth of quantum mechanics) is obtained by the Liouville transformation applied to the 1,1;2- $\mathfrak{I}$Ref CSLE with the so-called (Natanson G., 2017-19) 'simple-pole' density function [20-22]. Two other TSI CSLEs (0,1;1- and 0,2;2- $\mathfrak{I}$Ref in our classification scheme) can be obtained from the mentioned 1,0;1- and 2,2;0- $\mathfrak{I}$Ref CSLEs respectively by the reflection *z*→1-*z* which maps the interval [0, 1] onto itself (Ishkhanyan A, 2016) and thereby result in the same (*h*-PT and accordingly MR) TSI potentials after the corresponding Liouville transformations [23].

As recently advocated by Ishkhanyan and Krainov one can use the same 0,0;0- $\mathfrak{I}$Ref CSLE for the pair of the Poschl-Teller potentials and the 1,0;1- $\mathfrak{I}$Ref CSLE for two other potentials (often referred to (Newton RG, 1966) as potentials of the Eckart-class); however, the corresponding Liouville transformations should be also done on the positive infinite interval (1, +∞) in both cases so we again come to the four (not two (Ishkhanyan AM, 2018) Liouville potentials [23,25].

It is worth mentioning that both changes of variable z(x) = $-\sinh^2$ x or z(x) = $\cosh^2$ x in the 1,1;2- $\mathfrak{I}$Ref CSLE on the negative and positive infinite intervals originally introduced in (Dabrowska JW et al, 1988; Dutt R et al, 1988; Cooper F et al, 1987) to generate the h-PT potential turns out to be preferable since the corresponding eigenfunctions become expressible in terms of a finite orthogonal set of Romanovski-Jacobi-polynomials (Romanovski VI, 1929; Lesky PA, 1995-1996) which can be then used for constructing finite Exceptional Orthogonal Polynomial (EOP) sequences [32-35, 22].

For the pair of the Eckart-class potentials the quantization scheme using the same 0,0;0- $\mathfrak{I}$Ref CSLE has been utilized by Quesne 2012; who, in following (Cooper F 1987-2001), misleadingly referred to the radial Manning-Rosen potential as 'Eckart' potential [36,37]. It should be reminded to the reader that the potential introduced by Eckart, 1930 is nothing but Bose AK, 1964 another form of the RM potential obtained by applying the Liouville transformation to the 0,2;2- $\mathfrak{I}$Ref CSLE on the negative semi-axis [38,39].

It was originally proven in (Natanson G, 2011) and then more scrupulously analyzed in (Natanson G, 2016) that the ground-energy Eigen function of the ℑRef CSLE for any TP with a positive discriminant is accompanied by three other *basic* solutions such that their analytical continuations into the complex plane remain finite at any regular point of the ℑRef CSLE [10, 40]. In particular this is true for the TSI 1,1;2-and 1,0;1-ℑRef CSLEs associated with the *t*- and *h*-PT potentials. As a result, the Darboux-Crum transforms (CTs) of these potentials form the net specified by two series of Maya diagrams [41].

On other hand the TPs for the 0,0;0- and 2,0;2-ℑRef CSLEs associated with the TSI potentials of the Eckart class have the zero discriminants so the DCℑs of these potentials form the net unambiguously specified by a single series of Maya diagrams [12].

As mentioned above the main purpose of this paper is to prove that the non-TFI 1,0;2-ℑRef CSLE preserves its form under the Darboux-Crum transformations (DCTs) using as their seed functions two different pairs of the basic solutions. Though two sequential LDTs formally applied to the CSLE has been already discussed in (Schnizer WA et al ;1993-94) we refer the reader to Leeb's paper [13,14,42].

Where these double-step DTs were interpreted as an extension of the Darboux-Crum-Krein theory of SLEs; though mentioning Krein's name in this connection in both (Rudyak BV et al;1987) and (Gómez-Ullate D et al 2011) was an overkill. The distinction between the Crum Wronskian (CW) and the Krein determinant (KD) [15, 41, 43, 44]. Arises only for the CW of more than two seed solutions of the CSLE in question and Leeb never discussed the solutions of the 'double-step' SLEs which indeed require the computation of the CW of three seed functions.

To our knowledge, the rigorous extension of Krein's work to the theory of the DCTs of CSLEs was first formulated in our works [45, 16]. It was shown that the CW of seed solutions of a CSLE and the corresponding KD differ by either a half-integer or integer power of the density function (see Appendix C below for details). As expected, this factor disappears for the Schrödinger equation since the density function is identically equal to 1.

As illuminated in Appendix D the crucial difference between the CW and KD representations of reference polynomial fractions (RefPFs) and solutions of the corresponding CSLEs is that all the KDs are regular at the TP zeros for any even number of seed functions. Another advantage of the Krein representation is that the KD does not contain higher-order derivatives of seed solutions and as a result can be easily expressed in terms of the so-called 'polynomial determinants' (PDs). As proven in (Natanson G., 2018) (See also G. Natanson, 2023) for the more recent application of our formalism), the latter polynomials in the general case of 0,0;2-ℑRef CSLE obey the Heine-type differential equations [46-49]. With the exponent parameters dependent on polynomial degrees.

The crucial difference of the 1,0;2-ℑRef CSLE of our current interest, compared with the general case is that the exponent difference (ExpDiff) for the pole at the origin is energy-independent and as a result, it is shifted by each LDT. As a direct consequence of this anomaly PDs may vanish at z=0 so one first needs to remove the corresponding integer power of z to get a polynomial obeying the Heine-type differential equation.

Radial Schrödinger equation with general solution expressible in terms of a superposition of two hypergeometric series

It was demonstrated in that the general potential exactly solvable in terms of hypergeometric series has the form [11]:

$$ {}_1V(x;\lambda_o,\mu_o;T_2) \equiv {}_1\underset{\to}{V}[z(x;T_2);\lambda_o,\mu_o;T_2] $$

$$ = -[z'(x;T_2)]^2\,{}_1I^o[z(x;T_2);\lambda_o,\mu_o] - \tfrac{1}{2}\{z,x\} \qquad (7) $$

with

$$ {}_1I^o[z;\lambda_o,\mu_o] = \frac{1-\lambda_o^2}{4z^2} + \frac{1}{4(1-z)^2} + \frac{\mu_o^2-\lambda_o^2+1}{4z(1-z)} \qquad (8) $$

and the symbolic expression {z, x} standing for the ℑRef PF of the CSLE

$$ \left\{\frac{d^2}{dz^2} + {}_1I^o[z;\lambda_o,\mu_o] + {}_1\rho[z;T_K]\varepsilon\right\}\,{}_1\Phi[z\,;\lambda_o,\mu_o;T_K;\varepsilon] = 0 $$

and the Schwarzian derivative [50, 39]. The energy measurement point was chosen in such a way that the 0,0;2-ℑRef potential in question vanishes at the limit $x \to \infty$ which was achieved by setting to 0 the zero-energy ExpDiff for the pole of CSLE at z =1.

If the free term of the TP is equal to 0 then the density function in question has a single pole at the origin:

$$ \underset{\to}{\rho}[z;\underset{\to}{a},\underset{\to}{b}] = \frac{a\,z+\underset{\to}{b}}{4z(1-z)^2}\;(\underset{\to}{b}>0) \qquad (10) $$

and the one-dimensional (1D) potential (7) turns into the radial 1,0;2-ℑRef potential with the centrifugal barrier at the origin,

$$ \underset{\to}{V}(r;\underset{\to}{\lambda}_o,\underset{\to}{\mu}_o;\underset{\to}{a},\underset{\to}{b}) \equiv {}_1\underset{\to}{V}[\underset{\to}{z}(r;\underset{\to}{a},\underset{\to}{b});\underset{\to}{\lambda}_o,\underset{\to}{\mu}_o;\underset{\to}{a},\underset{\to}{b}] $$

$$ = -[\underset{\to}{z}'(r;\underset{\to}{a},\underset{\to}{b})]^2\,{}_1I^o[\underset{\to}{z}(r;\underset{\to}{a},\underset{\to}{b});\underset{\to}{\lambda}_o,\underset{\to}{\mu}_o] - \tfrac{1}{2}\{\underset{\to}{z},r\} \qquad (11) $$

where the change of variable z = $\underset{\to}{z}(r;\underset{\to}{a},\underset{\to}{b})$ is determined by ODE (4) with density function (10). (The MR potential associated with the limiting case $\underset{\to}{b}=0$ requires special consideration.)

The corresponding RefPF ${}_1I^o[\underset{\to}{z};\underset{\to}{\lambda}_o,\underset{\to}{\mu}_o]$ is defined via general formula (8), with the symbols $\underset{\to}{\lambda}_0$ and $\underset{\to}{\mu}_0$ used instead of λ_0 and μ_0 which are preserved by us for the ℑRef potentials on the line [10].

ODE (4) thus takes form:

$$ \underset{\to}{z}'(r;\underset{\to}{a},\underset{\to}{b}) = \frac{2\sqrt{\underset{\to}{z}}(1-\underset{\to}{z})}{\sqrt{\underset{\to}{a}\,\underset{\to}{z}+\underset{\to}{b}}} \qquad (12) $$

(with prime again standing for the derivative with respect to r) which is solved under the boundary condition

$$ \underset{\to}{z}(0;\underset{\to}{a},\underset{\to}{b}) = 0 \qquad (13) $$

In particular [1, 2] radial potential (9) turns into the *h*-PT potential if

$\overset{a}{\to}=0$ (K=1).
Setting

$$c_1 := \sqrt{T_2[1]} = \underset{\to}{a} + \underset{\to}{b} \qquad (14)$$

we can represent the change of variable $r[\underset{\to}{z}]$ as the sum of the two integrals

$$r[\underset{\to}{z}] = r_1[\underset{\to}{z}] + r_2[\underset{\to}{z}] \qquad (15)$$

where

$$r_1[\underset{\to}{z}] := \tfrac{1}{2} c_1 \int_0^{\underset{\to}{z}} \frac{dz}{(1-z)\sqrt{T_2[z]}} = \tfrac{1}{2}\sqrt{c_1}\, arcosh \frac{2c_1/(1-\underset{\to}{z}) - c_1 - \underset{\to}{a}}{\underset{\to}{b}} \qquad (16)$$

and

$$r_2[\underset{\to}{z}] := -\tfrac{1}{2}\underset{\to}{a} \int_0^{\underset{\to}{z}} \frac{dz}{\sqrt{z(\underset{\to}{a}\,z + \underset{\to}{b})}} = \begin{cases} \tfrac{1}{2}\sqrt{|\underset{\to}{a}|}\, arccos\,(1 - 2|\underset{\to}{a}|\underset{\to}{z}/\underset{\to}{b}) & \text{for } \underset{\to}{a} < 0, \\ -\tfrac{1}{2}\sqrt{\underset{\to}{a}}\, arccosh(1 + 2\underset{\to}{a}\,\underset{\to}{z}/\underset{\to}{b}) & \text{for } \underset{\to}{a} > 0. \end{cases} \qquad (17)$$

The Schwarzian derivative in question

$$\{\underset{\to}{z}, r\} = -\frac{2}{\underset{\to}{z}(\underset{\to}{a}\,\underset{\to}{z} + \underset{\to}{b})} - \frac{2(1-\underset{\to}{z})^2}{(\underset{\to}{a}\,\underset{\to}{z} + \underset{\to}{b})^2}\left[\underset{\to}{a} - \frac{2(\underset{\to}{a} + \underset{\to}{b})\underset{\to}{z} - \underset{\to}{b}}{\underset{\to}{z}(1-\underset{\to}{z})} - \frac{5}{4}\frac{\underset{\to}{b}^2}{\underset{\to}{z}(\underset{\to}{a}\,\underset{\to}{z} + \underset{\to}{b})}\right] \qquad (18)$$

can be directly obtained from the general formula derived in by setting the free term and discriminant of the TP to 0 and

$$\underset{\to}{\Delta}_T \equiv \underset{\to}{b}^2 \qquad (19)$$

Respectively [11]. Re-writing (18) as

$$\{\underset{\to}{z}, r\} = -\frac{3(1-\underset{\to}{z})^2}{2\underset{\to}{z}(\underset{\to}{a}\,\underset{\to}{z} + \underset{\to}{b})} - \frac{2}{\underset{\to}{a}\,\underset{\to}{z} + \underset{\to}{b}} + \frac{2\underset{\to}{a}(1-\underset{\to}{z})}{(\underset{\to}{a}\,\underset{\to}{z} + \underset{\to}{b})^2} - \frac{\underset{\to}{a}(1-\underset{\to}{z})^2(\underset{\to}{a}\,\underset{\to}{z} + 6\underset{\to}{b})}{2(\underset{\to}{a}\,\underset{\to}{z} + \underset{\to}{b})^3} \qquad (20)$$

we can represent the given radial potential as

$$\underset{\to}{V}[\underset{\to}{z}; \underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \underset{\to}{a}, \underset{\to}{b}] = \frac{(\underset{\to}{\lambda}_o^2 - \tfrac{1}{4})(1-\underset{\to}{z})}{\underset{\to}{z}(\underset{\to}{a}\,\underset{\to}{z} + \underset{\to}{b})} + \underset{\to}{V}[\underset{\to}{z}; \tfrac{1}{2}, \underset{\to}{\mu}_o; \underset{\to}{a}, \underset{\to}{b}] \qquad (21)$$

where the second term in the sum represents the radial potential with the zero centrifugal barrier

$$\underset{\to}{V}[\underset{\to}{z}; \tfrac{1}{2}, \underset{\to}{\mu}_o; \underset{\to}{a}, \underset{\to}{b}] \equiv \frac{1-\underset{\to}{z}}{\underset{\to}{a}\,\underset{\to}{z} + \underset{\to}{b}}\left[\underset{\to}{\lambda}_o^2 - \underset{\to}{\mu}_o^2 - \frac{\underset{\to}{a}}{\underset{\to}{a}z + \underset{\to}{b}} + \frac{\underset{\to}{a}(1-\underset{\to}{z})(\underset{\to}{a}\,\underset{\to}{z} + 6\underset{\to}{b})}{4(\underset{\to}{a}\,\underset{\to}{z} + \underset{\to}{b})^2}\right] \qquad (22)$$

As originally demonstrated by the author the reflection of the latter potential at the origin results in the symmetric potential re-discovered by Ginocchio [11, 49]. A decade later. In the following paper Ginocchio also constructed a solvable radial potential with a varying mass which turns into radial potential (21) in the constant-mass limit [3-5].

By choosing coefficient (14) equal to 1 we can represent density function (10) as

$$_1\underset{\to}{\rho}(\underset{\to}{z}; \kappa) = \frac{\kappa\underset{\to}{z} + 1 - \kappa}{4\underset{\to}{z}(1-\underset{\to}{z})^2} \qquad (23)$$

Where

$$\kappa := \underset{\to}{a}/c_1 \qquad (24)$$

We require that the simple zero of density function (23),

$$\underset{\to}{z}_T(\kappa) = 1 - 1/\kappa \qquad (25)$$

lies outside the interval [0, 1]. This requirement holds iff $\kappa < 1$, namely,

$$\underset{\to}{z}_T(\kappa) < 0 \text{ for } 0 \le \kappa < 1,$$
$$\underset{\to}{z}_T(\kappa) > 1 \text{ for } \kappa < 0. \qquad (26)$$

(For $\kappa = 0$ and $\kappa = 1$ the radial $\mathcal{J}$Ref potential turns into the h-PT and MR potentials respectively.)

The Principal Frobenius Solutions (PFS) near the origin can be represented for negative energies as

$$_1\Phi_0[\underset{\to}{z}; \underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; \varepsilon] = \sqrt{\underset{\to}{z}(1-\underset{\to}{z})}\, \underset{\to}{z}^{\tfrac{1}{2}\underset{\to}{\lambda}_o} (1-\underset{\to}{z})^{-\tfrac{1}{2}\sqrt{-\varepsilon}} \times$$
$$F[\alpha(\underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; -\sqrt{|\varepsilon|}), \beta(\underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; -\sqrt{|\varepsilon|}); \underset{\to}{\lambda}_o + 1; \underset{\to}{z}], \qquad (27)$$

where $\underset{\to}{\alpha}(\underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; \Xi)$ and $\underset{\to}{\beta}(\underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; \Xi)$ are two roots of the indicial equation

$$X^2 - (\underset{\to}{\lambda}_o + \Xi + 1)X + \tfrac{1}{4}(\underset{\to}{\lambda}_o + \Xi + 1)^2 - \mu^2(\underset{\to}{\mu}_o; \kappa\Xi^2) = 0 \qquad (28)$$

for the pole of the CSLE at infinity, with

$$\mu(\underset{\to}{\mu}_o; \delta) := \sqrt{\underset{\to}{\mu}_o^2 + \delta} \qquad (29)$$

i.e.,

$$\underset{\to}{\alpha}(\underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; \Xi) + \underset{\to}{\beta}(\underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; \Xi) = \underset{\to}{\lambda}_o + \Xi + 1. \qquad (30)$$

Also [11]

$$\underset{\to}{\beta}(\underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; \Xi) - \underset{\to}{\alpha}(\underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; \Xi) = \mu(\underset{\to}{\mu}_o; \kappa\Xi^2) \qquad (31)$$

if we choose

$$\underset{\to}{\alpha}(\underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; \Xi) < \underset{\to}{\beta}(\underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; \Xi) \qquad (32)$$

Combining (30) and (31) one finds

$$\underset{\to}{\alpha}(\underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; \Xi) = \tfrac{1}{2}[\underset{\to}{\lambda}_o + \Xi + 1 - \mu(\underset{\to}{\mu}_o; \kappa\Xi^2)],$$
$$\underset{\to}{\beta}(\underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; \Xi) = \tfrac{1}{2}[\underset{\to}{\lambda}_o + \Xi + 1 + \mu(\underset{\to}{\mu}_o; \kappa\Xi^2)] \qquad (33)$$

and therefore

$$\underset{\to}{\alpha}(\underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; \Xi)\underset{\to}{\beta}(\underset{\to}{\lambda}_o, \underset{\to}{\mu}_o; \kappa; \Xi) = \tfrac{1}{4}[(\underset{\to}{\lambda}_o + \Xi + 1)^2 - \mu^2(\underset{\to}{\mu}_o; \kappa\Xi^2)] \qquad (34)$$

Note that the sign of the parameter $\Xi = -\sqrt{-\varepsilon}$ in the right-hand side of (30) is selected in such a way that the corresponding hypergeometric function remains finite at both quantization ends 0 and 1 unless ε coincides with an eigenvalue of the given $\mathcal{J}$Ref CSLE.

It is crucial that the ExpDiff for the pole of the 1,0;2-$\mathcal{J}$Ref CSLE at the origin is energy- independent. Extending condition (20) in to an arbitrary 'quasi-rational' solution (q-RS), as it has been already done in for the 0,0;2-$\mathcal{J}$Ref CSLE, and setting [10,11].

$$c_0 = 0,\ h_0 + 1 = \underset{\to}{\lambda}_o^2,\ h_1 = -1,\ \underset{\to}{\mu}_o^2 = f + 1 \qquad (35)$$

brings us to the q-RSs

$$_1\underset{\to}{\phi}_{\mathbf{t}_\pm m}[\underset{\to}{z}; \underset{\to}{\lambda}_o, \underset{\to}{\lambda}_{1;\mathbf{t}_\pm m}] = \sqrt{\underset{\to}{z}(1-\underset{\to}{z})}\, \underset{\to}{z}^{\pm\tfrac{1}{2}\underset{\to}{\lambda}_o} (1-\underset{\to}{z})^{\tfrac{1}{2}\underset{\to}{\lambda}_{1;\mathbf{t}_\pm m}} P_m^{(\underset{\to}{\lambda}_{1;\mathbf{t}_\pm m},\, \underset{\to}{\lambda}_o)}(2\underset{\to}{z} - 1) \qquad (36)$$

at the energies

$$\varepsilon_{\mathbf{t}_{\pm}m} = -\lambda^2_{1;\mathbf{t}_{\pm}m} \qquad (37)$$

where the exponent parameters $\lambda_{1;\mathbf{t}_{\pm}m}$ are given by one of the roots of the quadratic equations [51, 52].

$$[\lambda_{1;\mathbf{t}_{\pm}m} \pm \lambda_o + 2m + 1]^2 = \mu_o^2 + \kappa\lambda^2_{1;\mathbf{t}_{\pm}m} \qquad (38)$$

Here, in following our early study [53]. On DTs of centrifugal-barrier potentials (long before the birth of the SUSY quantum mechanics, the labels **a** and **b** are used to identify principal Frobenius solutions (PFSs) near the lower and upper ends of the quantization interval [0, 1, 54-56], while the eigenfunctions and any q-RS irregular near both singular endpoints are specified by the labels **c** and **d**.

Note that Sukumar in his pioneering study on SUSY partner of radial potentials a nearly decade later arranged these four types of TFs in a slightly different order:

a = T_3, **b** = T_4, **c** = T_1, **d** = T_2,

while Ancarani and Baye [57]. Replaced Sukumar's original notations by the more mnemonic notations

$$T_+^+ = T_2 = \mathbf{c}, T_-^- = T_1 = \mathbf{d}, T_3 = T_-^0 = \mathbf{a}, T_4 = T_+^0 = \mathbf{b}, \qquad (39)$$

with the superscript indicating whether the ground-state energy is raised (+), lowered () or left unchanged under action of the DT using the given solution as the TF (while the subscript signifies whether the phase shift is increased -(+) or decreased (-) in this case.) In Quesne's [58, 59, 36].

Present-time classification scheme of quasi-rational TFs the Cases I, II, and III correspond to the types **a**, **b**, and **d** respectively. An examination of q-RSs (36) reveals that

$$\mathbf{t}_+ = \mathbf{a} \text{ or } \mathbf{c},\ \mathbf{t}_- = \mathbf{b} \text{ or } \mathbf{d}. \qquad (40)$$

Representing quadratic equations (38) in the standard form

$$(1-\kappa)\lambda^2_{1;\mathbf{t}_{\pm},m} + 2(\pm\lambda_o + 2m+1)\lambda_{1;\mathbf{t}_{\pm},m} + (2m+1\pm\lambda_o)^2 - \mu_o^2 = 0$$

(41)

we find that the linear coefficient of the quadratic equation for the q-RSs $\mathbf{t}_{+},m$ is necessarily positive and therefore the equation has two negative roots if both its leading coefficient and free term are positive. Since $\kappa < 1$ the potential may have at least n+1 bound energy levels iff the given quadratic equation has a negative free term

$$\mu_o^2 > (\lambda_o + 2n + 1)^2 \qquad (42)$$

If condition (42) holds then the leading coefficient and free term of each quadratic equation (41) for the two q-RSs $\mathbf{t}\_m$ with m=n also have opposite signs so both quadratic equations have positive discriminants

$$\Delta_n(\pm\lambda_o, \mu_o; \kappa) \equiv 4[\kappa(2n+1\pm\lambda_o)^2 + (1-\kappa)\mu_o^2] \qquad (43)$$

We thus proved that the roots of each quadratic equation must have opposite sign for m = n if the potential has at least n+1 bound energy levels and therefore that the eigenfunction **c**,n of the n[th] excited energy state is accompanied by the three q-RSs: **a**,n , **b**,n , and **d**,n such that

$$\lambda_{1;\mathbf{c}n}(\lambda_o, \mu_o; \kappa) = \frac{\tfrac{1}{2}\sqrt{\Delta_n(\lambda_o, \mu_o; \kappa)} - \lambda_o - 2n - 1}{1-\kappa} > 0, \quad (44)$$

$$\lambda_{1;\mathbf{a}n}(\lambda_o, \mu_o; \kappa) = -\frac{\tfrac{1}{2}\sqrt{\Delta_n(\lambda_o, \mu_o; \kappa)} + \lambda_o + 2n + 1}{1-\kappa} < 0$$

And

$$\lambda_{1;\mathbf{d}n}(\lambda_o, \mu_o; b) = -\frac{\tfrac{1}{2}\sqrt{\Delta_n(-\lambda_o, \mu_o; b)} - \lambda_o + 2n + 1}{1-b} < 0$$

$$\lambda_{1;\mathbf{b}n}(\lambda_o, \mu_o; \kappa) = \frac{\tfrac{1}{2}\sqrt{\Delta_n(-\lambda_o, \mu_o; \kappa)} + \lambda_o - 2n - 1}{1-\kappa} > 0 \quad (45)$$

in agreement with the general theorem proven in for the density function formed by a TP with a positive discriminant [10].

As a direct consequence of this proof we conclude that the 1,0;2-$\mathcal{J}$Ref CSLE has a quartet of basic solutions of distinct types **a**, **b**, **c**, and **d** (n=0) if it has at least one solution normalizable with the weight (23).

In principle any combination of the ground-energy eigenfunction **c**, 0 and all the q-RSs of the types **a** or **b** below the ground energy level can be used as seed functions to construct the DC net of the exactly solvable radial potentials. In addition, one can add juxtaposed pairs of eigenfunctions based on the extension of the Adler theorem proven by us in for the 0,0;2-$\mathcal{J}$Ref CSLE but easily extended to its counter-part with the energy-independent ExpDiff at the origin [46, 60].

In this paper we focus solely on the double-step DC transformations (DCTs) using the pair of the basic solutions **a**, 0 and **c**, 0 or **b**, 0 and **d**, 0 .

While the first combination simply represents one of the first DCTs starting the aforementioned DC net, the combination **b**, 0 and **d**, 0 is only admissible if the PFS of the CSLE near the upper singular endpoint at the energy $\varepsilon = \varepsilon_{\mathbf{b},0}$ lies below the q-RS **d**, 0 , i.e.,

$$\varepsilon_{\mathbf{b},0} < \varepsilon_{\mathbf{d},0}. \qquad (46)$$

Comparing the first of exponent parameters (45) with the absolute value of the second shows that the latter constraint holds iff $\lambda_o > 1$. We shall come back to this issue in next Section after proving that DCTs of the 1,0;2-$\mathcal{J}$Ref CSLE using the cited pairs of the basic solutions as seed functions bring us back to the Fuschian CSLE with three poles while shifting by 2 the ExpDiff for the pole at the origin. Bearing in mind that the DCTs in question keep unchanged density function (23) this implies that the 1,0;2-$\mathcal{J}$Ref CSLE preserves its form under both second-order DCTs (with $\lambda_o > 1$ in the second instance) which is the main purpose of this paper.

Note that the pairs of solutions used by us as the seed functions for the corresponding second-order DCTs represent the very special cases of the suppression and respectively addition of the ground-energy state introduced by Baye as a systematic method for constructing phase-equivalent potentials so the mentioned shifts of the ExpDiffs by 2 are direct consequences of his Eqs. (3.1) and (3.6) [61].

Double-step form invariance of the 1,0;2-$\mathfrak{I}$Ref CSLE

Let us now consider the double-step DCT using a pair of basic seed functions

$$ {}_1\underline{\phi}_{\pm,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa] = \underline{z}^{\frac{1}{2}(\lambda+1)}(1-\underline{z})^{\frac{1}{2}[1+\underline{\lambda}_{\pm,0}(\lambda,\underline{\mu}_o;\kappa)]}, \quad (47) $$

where λ is $\underline{\lambda}_o$ or $-\underline{\lambda}_o$ and

$$ \underline{\lambda}_{-,0}(\underline{\lambda}_o,\underline{\mu}_o;\kappa) = \underline{\lambda}_{1;\mathbf{a},0}(\underline{\lambda}_o,\underline{\mu}_o;\kappa), $$
$$ \underline{\lambda}_{+,0}(-\underline{\lambda}_o,\underline{\mu}_o;\kappa) = \underline{\lambda}_{1;\mathbf{b},0}(\underline{\lambda}_o,\underline{\mu}_o;\kappa), $$
$$ \underline{\lambda}_{+,0}(\underline{\lambda}_o,\underline{\mu}_o;\kappa) = \underline{\lambda}_{1;\mathbf{c},0}(\underline{\lambda}_o,\underline{\mu}_o;\kappa), $$
$$ \underline{\lambda}_{-,0}(-\underline{\lambda}_o,\underline{\mu}_o;\kappa) = \underline{\lambda}_{1;\mathbf{d},0}(\underline{\lambda}_o,\underline{\mu}_o;\kappa). \quad (48) $$

The RefPF for the transformed CSLE can be represented as [42]

$$ \underline{I}^o[\underline{z};\lambda,\underline{\mu}_o;\kappa\,|+,0;-,0] = I^o[\underline{z};\lambda,\underline{\mu}_o] + 2\sqrt{{}_1\underline{\rho}[\underline{z};\kappa]}\,\frac{d}{d\underline{z}}\frac{ld\,{}_1\underline{W}[\underline{z};\lambda,\underline{\mu}_o;\kappa]}{\sqrt{{}_1\underline{\rho}[\underline{z};\kappa]}} \quad (49) $$

Where

$$ {}_1\underline{W}[\underline{z};\lambda,\underline{\mu}_o;\kappa] \equiv W\{{}_1\underline{\phi}_{+,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa],\,{}_1\underline{\phi}_{-,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa]\} = \begin{vmatrix} {}_1\underline{\phi}_{+,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa] & {}_1\underline{\phi}_{-,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa] \\ {}_1\dot{\underline{\phi}}_{+,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa] & {}_1\dot{\underline{\phi}}_{-,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa] \end{vmatrix}. \quad (50) $$

Substituting (47) into (50) shows that the Wronskian of the selected pair of the basic solutions is given by the simple formula

$$ {}_1\underline{W}[\underline{z};\lambda,\underline{\mu}_o;\kappa] = \frac{\underline{\lambda}_{+,0}(\lambda,\underline{\mu}_o;\kappa) - \underline{\lambda}_{-,0}(\lambda,\underline{\mu}_o;\kappa)}{2(1-\underline{z})}]\ {}_1\underline{\phi}_{+,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa]\,{}_1\underline{\phi}_{-,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa]) \quad (51) $$

which represents the core of our derivation. Combining it with the general formula for the derivative of the Wronskian of two solutions of the generic CSLE

$$ {}_1\dot{\underline{W}}[\underline{z};\lambda,\underline{\mu}_o;\kappa] = \begin{vmatrix} {}_1\underline{\phi}_{+,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa] & {}_1\underline{\phi}_{-,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa] \\ {}_1\ddot{\underline{\phi}}_{+,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa] & {}_1\ddot{\underline{\phi}}_{-,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa] \end{vmatrix} $$
$$ = \left[\underline{\varepsilon}_+(\lambda,\underline{\mu}_o;\kappa) - \underline{\varepsilon}_-(\lambda,\underline{\mu}_o;\kappa)\right]{}_1\underline{\rho}[\underline{z};\kappa]\ {}_1\underline{\phi}_{+,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa]\,{}_1\underline{\phi}_{-,0}[\underline{z};\lambda,\underline{\mu}_o;\kappa] \quad (52) $$

one can represent the logarithmic derivative of Wronskian (50) as

$$ ld\,{}_1\underline{W}[\underline{z};\lambda,\underline{\mu}_o;\kappa] = -\,[\underline{\lambda}_{+,0}(\lambda,\underline{\mu}_o;\kappa) + \underline{\lambda}_{-,0}(\lambda,\underline{\mu}_o;\kappa)]\frac{\kappa[\underline{z}-\underline{z}_T(\kappa)]}{2\underline{z}(1-\underline{z})}, \quad (53) $$

where we also used (37) to express the energy difference in terms of exponent parameters (48):

$$ \underline{\varepsilon}_\pm(\lambda,\underline{\mu}_o;\kappa) := -\underline{\lambda}^2_{\pm,0}(\lambda,\underline{\mu}_o;\kappa). \quad (54) $$

Taking advantage of the fact that $\underline{\lambda}_{\pm,0}(\lambda,\underline{\mu}_o;\kappa)$ are the two roots of the quadratic equation

$$ (1-\kappa)\underline{\lambda}^2_{\pm,0}(\lambda,\underline{\mu}_o;\kappa) + 2(\lambda+1)\underline{\lambda}_{\pm,0}(\lambda,\underline{\mu}_o;\kappa) + (1+\lambda)^2 - \underline{\mu}_o^2 = 0 \quad (55) $$

so

$$ \underline{\lambda}_{+,0}(\lambda,\underline{\mu}_o;\kappa) + \underline{\lambda}_{-,0}(\lambda,\underline{\mu}_o;\kappa) = \frac{2(\lambda+1)}{\kappa-1} \quad (56) $$

we can re-write (53) as

$$ ld\,{}_1\underline{W}[\underline{z};\lambda,\underline{\mu}_o;\kappa] = -\frac{\lambda+1}{\kappa-1}\times\frac{\kappa[\underline{z}-\underline{z}_T(\kappa)]}{\underline{z}(1-\underline{z})}. \quad (57) $$

with the numerator of the z-dependent fraction is positive for $0 \le \underline{z} \le 1$. Consequently

$$ \frac{ld\,{}_1\underline{W}[z;\lambda,\underline{\mu}_o;\kappa]}{\sqrt{{}_1\underline{\rho}[z;\kappa]}} = \frac{2(\lambda+1)}{\kappa\,\underline{z}_T(\kappa)}\sqrt{\kappa[z-\underline{z}_T(\kappa)]/z}. \quad (58) $$

Substituting (58) into the second summand in (49) and taking into account that

$$ \frac{d}{dz}\sqrt{\kappa[1-\underline{z}_T(\kappa)/z]} = \frac{\kappa\,\underline{z}_T(\kappa)}{2z^{3/2}\sqrt{\kappa[z-\underline{z}_T(\kappa)]}} \quad (59) $$

thus gives

$$ 2\sqrt{{}_1\underline{\rho}[z;\kappa]}\,\frac{d}{dz}\frac{ld\,{}_1\underline{W}[z;\lambda,\underline{\mu}_o;\kappa]}{\sqrt{{}_1\underline{\rho}[z;\kappa]}} = \frac{\lambda+1}{z^2(1-z)}. \quad (60) $$

This brings us to the sought-for relation

$$ \underline{I}^o[z;\lambda,\underline{\mu}_o\,|+,0;-,0] = I^o[z;\lambda+2,\underline{\mu}_o] + \frac{\lambda+1}{z(1-z)}. \quad (61) $$

for the RefPF of the transformed CSLE. As a direct consequence of (61) we assert that the two Liouville potentials associated with the RefPFs in question are related via the translational shape-invariance condition

$$ \underline{V}[\underline{z}(r);\lambda,\underline{\mu}_o;\kappa\,|+,0;-,0] = \underline{V}[\underline{z}(r);\lambda+2,\underline{\mu}_o;\kappa] - 4(\lambda+1). \quad (62) $$

with $\lambda = \pm\underline{\lambda}_o$ outside the interval [-2, 0].

If we use the pair of seed functions of the types **a** and **c** then the DCT in question increases the ExpDiff for the pole at the origin by 2, while erasing the ground energy level. Examination of quadratic equation (41) for the q-RSs $\underline{\mathbf{t}}_+, m$ shows that

$$ \underline{\lambda}_{1;\mathbf{c},n-1}(\lambda_o+2,\mu_o;\kappa) = \underline{\lambda}_{1;\mathbf{c},n}(\lambda_o,\mu_o;\kappa). \quad (63) $$

We thus confirmed that the discrete energy spectrum of the transformed potential $V(r;\lambda_o+2,\mu_o;\kappa)$ is formed by n_o-1 excited energy levels of radial potential (11), in agreement with the conventional rules of the SUSY quantum mechanics [21].

The problem becomes more complicated for the second pair of seed functions of the types **b** and **d** so the corresponding double-step DCT inserts the new energy level $\varepsilon_{1;\mathbf{c}0}(\lambda_o-2,\mu_o;\kappa)$ while decreasing the ExpDiff for the pole at the origin by 2 iff $\lambda_o>2$. As far as the above constrains holds n_o excited energy levels of the transformed potential $V(r;\lambda_o-2,\mu_o;\kappa)$ precisely match the discrete energy spectrum of original radial potential (11):

$$\varepsilon_{\mathbf{c},n+1}(\lambda_o-2,\mu_o;\kappa)=\varepsilon_{\mathbf{c},n}(\lambda_o,\mu_o;\kappa) \quad \text{for } \lambda_o>2 \quad (64)$$

as anticipated.

However, the discrete energy spectrum of the potential $V(r;2-\lambda_o,\mu_o;\kappa)$ for $0<\lambda_o<1$ or $1<\lambda_o<2$ is given by the different formulas

$$\varepsilon_{\mathbf{c},n}(2-\lambda_o,\mu_o;\kappa)=-\lambda^2_{1;\mathbf{c},n}(2-\lambda_o,\mu_o;\kappa)\neq\varepsilon_{\mathbf{c},n}(\lambda_o,\mu_o;\kappa) \quad (65)$$

In the region $0<\lambda_o<1$ the 1,0;2-$\mathfrak{J}$Ref CSLE has a limit-circle (LC) singularity at the origin. As a result the DTs with the TFs irregular at this singular point violate conventional rules of the SUSY quantum mechanics [37]. If $1<\lambda_o<2$ then the RDT of CSLE (9) with the TF **b**,0 results in the isospectral radial potential $V(r;\lambda_o,\mu_o;\kappa\,|\,\mathbf{b},0)$ of the Heun class exactly quantized by polynomial solutions of the Heun equation [25,62]. However, the DT in question reduces by 1 the ExpDiff for the pole at the origin and thereby places this singular point into the LC region of the CSLE with the RefPF for.

As a result, any DT using a TF irregular at origin generates a potential with all bound energy levels essentially different from those for the mentioned radial potential of the Heun class. The double-step DCT with seed solutions **b**,0 and **d**,0 thus generates the 1,0;2-$\mathfrak{J}$Ref potential with the energy levels having no resemblance with the original energy spectrum.

Making use of Schulze-Halberg's the general formula for the solutions of the generic CSLE constructed using an arbitrary DCT we can represent the DC$\mathfrak{J}$s of PFS (27) under action of the double-step DCTs of our interest as [63]

$$_1\Phi_0[z;\lambda_o,\mu_o;\kappa;\varepsilon\,|\,\mathbf{t}_\pm,0;\mathbf{t}'_\pm,0] \quad (66)$$

$$=\frac{\mathrm{Wr}\{_1\phi_{\mathbf{t}_\pm,0}[z;\lambda_o,\mu_o;\kappa],\,_1\phi_{\mathbf{t}'_\pm,0}[z;\lambda_o,\mu_o;\kappa],\,_1\Phi_0[z;\lambda_o,\mu_o;\kappa;\varepsilon]\}}{_1\rho[z;\kappa]\,W[z;\lambda_o,\mu_o;\kappa\,|\,\mathbf{t}_\pm,0;\mathbf{t}'_\pm,0]}$$

$(\mathbf{t}_\pm\neq\mathbf{t}'_\pm)$,

where

$$W[z;\lambda_o,\mu_o;\kappa\,|\,\mathbf{t}_\pm,0;\mathbf{t}'_\pm,0]$$

$$:=\mathrm{Wr}\{_1\phi_{\mathbf{t}_\pm,0}[z;\lambda_o,\mu_o;\kappa],\,_1\phi_{\mathbf{t}'_\pm,0}[z;\lambda_o,\mu_o;\kappa]\}$$

$$=(\lambda_{1;\mathbf{t}'_\pm,0}-\lambda_{1;\mathbf{t}_\pm,0})\,_1\phi_{\mathbf{t}_\pm,0}[z;\lambda_o,\mu_o;\kappa]\,_1\phi_{\mathbf{t}'_\pm,0}[z;\lambda_o,\mu_o;\kappa]/(1-z). \quad (67)$$

Converting the CWs to the KDs via (C1) with j = ℓ = 1 thus gives

$$_1\Phi_0[z;\lambda_o,\mu_o;\kappa;\varepsilon\,|\,\mathbf{t}_\pm,0;\mathbf{t}'_\pm,0]=\frac{_1\mathcal{K}_0[z;\lambda_o,\mu_o;\kappa;\varepsilon\,|\,\mathbf{t}_\pm,0;\mathbf{t}'_\pm,0]}{W[z;\lambda_o,\mu_o;\kappa\,|\,\mathbf{t}_\pm,0;\mathbf{t}'_\pm,0]} \quad (68)$$

where

$$_1\mathcal{K}_0[z;\lambda_o,\mu_o;\kappa;\varepsilon\,|\,\mathbf{t}_\pm,0;\mathbf{t}'_\pm,0]:= \quad (69)$$

$$\begin{vmatrix} _1\phi_{\mathbf{t}_\pm,0}[z;\lambda_o,\mu_o;\kappa] & _1\phi_{\mathbf{t}'_\pm,0}[z;\lambda_o,\mu_o;\kappa] & _1\Phi_0[z;\lambda_o,\mu_o;\kappa;\varepsilon] \\ _1\dot{\phi}_{\mathbf{t}_\pm,0}[z;\lambda_o,\mu_o;\kappa] & _1\dot{\phi}_{\mathbf{t}'_\pm,0}[z;\lambda_o,\mu_o;\kappa] & _1\dot{\Phi}_0[z;\lambda_o,\mu_o;\kappa;\varepsilon] \\ \lambda^2_{1;\mathbf{t}_\pm,0}\,_1\phi_{\mathbf{t}_\pm,0}[z;\lambda_o,\mu_o;\kappa] & \lambda^2_{1;\mathbf{t}'_\pm,0}\,_1\phi_{\mathbf{t}'_\pm,0}[z;\lambda_o,\mu_o;\kappa] & -\varepsilon\,_1\Phi_0[z;\lambda_o,\mu_o;\kappa;\varepsilon] \end{vmatrix}.$$

Note the disappearance of the density function from the denominator of fraction (68), in contrast with (66). This is the main advantage of the KD representation compared with the Crum formula. As explicitly demonstrated below the numerator of fraction (68) and therefore the fraction itself do not have a pole at the TP zero $z_T(\kappa)$ as expected from the fact that RefPF (61) is regular at this point.

Setting

$$_1\Theta[z;\lambda_o,\mu_o;\kappa;\varepsilon\,|\,\mathbf{t}_\pm,0;\mathbf{t}'_\pm]:=$$

$$\frac{\vartheta[z;\pm\lambda_o,\lambda_{1;\mathbf{t}_\pm,0}]\,\vartheta[z;\pm\lambda_o,\lambda_{1;\mathbf{t}'_\pm,0}]}{z(1-z)}\,\vartheta[z;\lambda_o,-\sqrt{-\varepsilon}]$$

With

$$\vartheta[z;\lambda_0,\lambda_1]:=z^{\frac{1}{2}(\lambda_0+1)}(1-z)^{\frac{1}{2}(\lambda_1+1)} \quad (71)$$

we can represent the numerator of fraction (68) as

$$_1\mathcal{K}_0[z;\lambda_o,\mu_o;\kappa;\varepsilon\,|\,\mathbf{t}_\pm,0;\mathbf{t}'_\pm,0]=\,_1\Theta[z;\lambda_o,\mu_o;\kappa;\varepsilon\,|\,\mathbf{t}_\pm,0;\mathbf{t}'_\pm,0]\times$$
$$_1F_0[z;\lambda_o,\mu_o;\kappa;\varepsilon\,|\,\mathbf{t}_\pm,0;\mathbf{t}'_\pm,0],$$

where the function

$$_1F_0[z;\lambda_o,\mu_o;\kappa;\varepsilon\,|\,\mathbf{t}_\pm,0;\mathbf{t}'_\pm,0]:=$$
$$_1\mathbf{D}\left\{-\sqrt{-\varepsilon}\,|\,\mathbf{t}_\pm,;\mathbf{t}'_\pm\right\}F[\alpha(\lambda_o,\mu_o;\kappa;-\sqrt{|\varepsilon|}),$$
$$\beta(\lambda_o,\mu_o;\kappa;-\sqrt{|\varepsilon|});\lambda_o+1;z]$$

will be termed the 'Darboux-Crum-Krein transform' (DCK$\mathfrak{J}$) of the hypergeometric function. Similarly we refer the first-order differential operators

$$_1\mathbf{D}\left\{-\sqrt{-\varepsilon}\,|\,\mathbf{t}_+;\mathbf{t}'_+\right\}$$

$$=-\frac{1}{2}z\begin{vmatrix} 1 & 1 & 1 \\ \lambda_{1;\mathbf{t}_+,0}+1 & \lambda_{1;\mathbf{t}'_+,0}+1 & 1-\sqrt{\varepsilon}+2(z-1)\frac{d}{dz} \\ -\lambda^2_{1;\mathbf{t}_+,0} & -\lambda^2_{1;\mathbf{t}'_+,0} & \varepsilon \end{vmatrix}$$

And

$${}_1\underline{\mathbf{D}}\left\{-\sqrt{-\varepsilon}\,|\,\underline{\mathbf{t}}_-;\underline{\mathbf{t}}'_-\right\}$$

$$= -\tfrac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ \underline{\lambda}_{1;\underline{\mathbf{t}}_-,0}\, z & \underline{\lambda}_{1;\underline{\mathbf{t}}'_-,0}\, z & -\sqrt{\varepsilon}\, z + 2(z-1)\left[\underline{\lambda}_o + z\frac{d}{dz}\right] \\ -\underline{\lambda}^2_{1;\underline{\mathbf{t}}_-,0} & -\underline{\lambda}^2_{1;\underline{\mathbf{t}}'_-,0} & \varepsilon \end{vmatrix}$$

(75)

as 'DCK generators'. It can be easily verify that the matrix expressions for the DCK generators can be simplified as follows

$${}_1\underline{\mathbf{D}}\left\{-\sqrt{-\varepsilon}\,|\,\underline{\mathbf{t}}_+;\underline{\mathbf{t}}'_+\right\} = -\tfrac{1}{2}\underline{z}(\underline{\lambda}_{1;\underline{\mathbf{t}}'_+,0} - \underline{\lambda}_{1;\underline{\mathbf{t}}_+,0})\times$$

$$\left[(\sqrt{-\varepsilon}+\underline{\lambda}_{1;\underline{\mathbf{t}}_+,0})(\sqrt{-\varepsilon}+\underline{\lambda}_{1;\underline{\mathbf{t}}'_+,0}) + 2(\underline{\lambda}_{1;\underline{\mathbf{t}}'_+,0}+\underline{\lambda}_{1;\underline{\mathbf{t}}_+,0})(\underline{z}-1)\frac{d}{d\underline{z}}\right]$$

And

$${}_1\underline{\mathbf{D}}\left\{-\sqrt{-\varepsilon}\,|\,\underline{\mathbf{t}}_-;\underline{\mathbf{t}}'_-\right\} = \tfrac{1}{2}(\underline{\lambda}_{1;\underline{\mathbf{t}}'_-,0} - \underline{\lambda}_{1;\underline{\mathbf{t}}_-,0})\times$$

$$\left\{(\sqrt{-\varepsilon}+\underline{\lambda}_{1;\underline{\mathbf{t}}'_-,0})(\sqrt{-\varepsilon}+\underline{\lambda}_{1;\underline{\mathbf{t}}_-,0})\underline{z} + 2(\underline{\lambda}_{1;\underline{\mathbf{t}}'_-,0}+\underline{\lambda}_{1;\underline{\mathbf{t}}_-,0})(1-\underline{z})\left(\underline{\lambda}_o + \underline{z}\frac{d}{d\underline{z}}\right)\right\}.$$

Bearing in mind that $\underline{\lambda}_{1;\underline{\mathbf{t}}_\pm,0}$ and $\underline{\lambda}_{1;\underline{\mathbf{t}}'_\pm,0}$ represent two roots (44) or (45) of the one of quadratic equation (41) with m=0, coupled with (29) we can represent the quadratic polynomials in $\sqrt{-\varepsilon}$ as

$$(\sqrt{-\varepsilon}+\underline{\lambda}_{1;\underline{\mathbf{t}}'_\pm,0})(\sqrt{-\varepsilon}+\underline{\lambda}_{1;\underline{\mathbf{t}}_\pm,0})$$

$$= \mu^2(\underline{\mu}_o, -\kappa\varepsilon) - (1 \pm \underline{\lambda}_o - \sqrt{-\varepsilon})^2$$

Let us now confirm that function (73) generated by operator (76) has the simple zero at the origin as expected from the fact that the DCT in question simply increases by 2 the ExpDiff for the pole of the CSLE at this point. First comparing (78) with (34) shows that

$$(\sqrt{-\varepsilon}+\underline{\lambda}_{1;\underline{\mathbf{t}}'_+,0})(\sqrt{-\varepsilon}+\underline{\lambda}_{1;\underline{\mathbf{t}}_+,0})$$

$$= -4\underline{\alpha}(\underline{\lambda}_o,\underline{\mu}_o;\kappa;-\sqrt{-\varepsilon})\,\underline{\beta}(\underline{\lambda}_o,\underline{\mu}_o;\kappa;-\sqrt{-\varepsilon}).$$

Taking into account (44) we can thus re-write operator (76) as

$${}_1\underline{\mathbf{D}}\left\{-\sqrt{-\varepsilon}\,|\,\underline{\mathbf{t}}_+;\underline{\mathbf{t}}'_+\right\} = -\frac{\sqrt{\underline{\Delta}_0(\underline{\lambda}_o,\underline{\mu}_o;\kappa)}}{2(1-\kappa)}\underline{z}\times \quad (80)$$

$$\left[\begin{matrix} -4\underline{\alpha}(\underline{\lambda}_o,\underline{\mu}_o;\kappa;-\sqrt{-\varepsilon})\,\underline{\beta}(\underline{\lambda}_o,\underline{\mu}_o;\kappa;-\sqrt{-\varepsilon}) \\ +4(\underline{\lambda}_o+1)(\underline{z}-1)\dfrac{d}{d\underline{z}} \end{matrix}\right].$$

Finally representing contiguous relation (13) in §33 in [64] as

$$\underline{\lambda}_o + 1(1-\underline{z})\frac{d}{d\underline{z}}F[\underline{\alpha},\underline{\beta};\underline{\lambda}_o+1;\underline{z}] = \underline{\alpha}\,\underline{\beta}\,F[\underline{\alpha},\underline{\beta};\underline{\lambda}_o+1;\underline{z}] + \quad (85)$$

$$(\underline{\alpha}-\underline{\lambda}_o-1)(\underline{\beta}-\underline{\lambda}_o-1)\left\{F[\underline{\alpha},\underline{\beta};\underline{\lambda}_o+2;\underline{z}] - F[\underline{\alpha},\underline{\beta};\underline{\lambda}_o+1;\underline{z}]\right\}$$

and setting α=β=γ=0 in (48) in [65]:

$$F[\underline{\alpha},\underline{\beta};\underline{\lambda}_o+1;\underline{z}] - F[\underline{\alpha},\underline{\beta};\underline{\lambda}_o+2;\underline{z}]$$

$$= \frac{\underline{\alpha}\underline{\beta}}{(\underline{\lambda}_o+1)(\underline{\lambda}_o+2)}zF[\underline{\alpha}+1,\underline{\beta}+1;\underline{\lambda}_o+3;\underline{z}]$$

Gives

$${}_1F_0[\underline{z};\underline{\lambda}_o,\underline{\mu}_o;\kappa;\varepsilon\,|\,\underline{\mathbf{t}}_-,0;\underline{\mathbf{t}}'_-,0] \quad (83)$$

$$\propto zF[\alpha(\underline{\lambda}_o,\underline{\mu}_o;\kappa;-\sqrt{|\varepsilon|})+1,$$

$$\beta(\underline{\lambda}_o,\underline{\mu}_o;\kappa;-\sqrt{|\varepsilon|})+1;\underline{\lambda}_o+3;0]$$

which completes the proof.

In closing let us also point to the fact that function (73) generated by operator (81) remains finite at the origin:

$${}_1F_0[0;\underline{\lambda}_o,\underline{\mu}_o;\kappa;\varepsilon\,|\,\underline{\mathbf{t}}_-,0;\underline{\mathbf{t}}'_-,0]$$

$$= 2\underline{\lambda}_o(\underline{\lambda}_o-1)\frac{\sqrt{\underline{\Delta}_0(-\underline{\lambda}_o,\underline{\mu}_o;\kappa)}}{1-\kappa}$$

as expected from the fact that the DCT in question simply decreases by 2 the ExpDiff for the pole of the CSLE at the origin while keeping the parameter $\underline{\mu}_o$ unchanged.

Note that all the results were obtained by us with no reference to the Liouville transformations leading to its quantum-mechanical applications. To be able to apply the developed formalism to the quantum-mechanical problems one simply needs to convert the 1,0;2-$\mathcal{J}$Ref CSLE to the Schrödinger equation with rational potential (11) using the change of variable (12) as it has been done in a slightly modified form [66-69].

CONCLUSION

It was shown that the radial potential exactly solvable via a superposition of two hypergeometric series (termed 'radial Jacobi-reference potential in the paper) has two pairs of q-RSs such that their analytical continuations do not have zeros at any regular point in the complex plane. Taking into account that the absolute values of the characteristic exponents (ChExps) for the pole at the origin are the same for all four 'basic' solutions the latter were grouped into two pairs via the requirement that the paired solutions share the same ChExp for the mentioned singularity. Each pair of the basic solutions **a**,0 and **c**,0 or **b**,0 and **d**,0 is then used as seed functions for the double-step DCT of the radial potential in question. It is proven that both transformation simply shifts by the ExpDiff for the pole of the CSLE at the origin while keeping unchanged the last two of the three potential parameters $\underline{\lambda}_o$, $\underline{\mu}_o$, and κ

Since the double-step shape invariance of the potential is the direct consequence of translational form invariance of the 1,0,2-$\mathcal{J}$Ref CSLE under two sequential LDTs it must retain if the Liouville transformation is performed on the infinite interval (+1, +∞) obtained from the finite interval (0, +1) by the reciprocal transformation $z \to 1/z$ [25] (assuming that the TP zero lies on the negative semi-axis). The latter transformation results in CSLE (9) with the density function:

$${}_1\rho[z;T_1] = \frac{T_1[z]}{4z^2(1-z)^2}$$

Where

$$T_1[0] > 0$$

The corresponding 'Linear Tangent Polynomial' (LTP) potential should be double-step shape invariant as well. If the TP $T_1[z]$ has a negative root then the Liouville transformation of the $\mathscr{J}\mathrm{Ref}$ CSLE with density function (85) can be again performed on the positive finite and positive infinite quantization intervals: (0, 1) and (1, ∞) accordingly as it was originally done by Quesne for the TSI limits of the Liouville potentials of the Eckart class: the RM (or Eckart) and MR potentials.

Note that Ishkhanyan and Krainov referred to the density functions ${}_1\rho[z]$ associated with TFI $\mathscr{J}\mathrm{Ref}$ CSLEs as 'discretization of Natanzon potentials' and correctly stressed that there were two pairs of the density functions such that the functions in each pair are related via linear fractional transformation. However, based on the cited observation they then erroneously claimed that there were only two potentials solvable by hypergeometric functions, contrary to the common knowledge that there are four potentials satisfying the latter requirement (t- and h-versions of the PT potential and already mentioned pair of the RM and MR potentials). The cited authors were apparently confused by the fact that the Liouville transformations for the potentials in each pair are performed on the finite and infinite intervals of the variable z separated by a singular end point and as a results associated with two completely changes of variable x(z).

In this communication we skip the discussion of the first step – the LDT resulting in rational potentials exactly quantized by polynomial solutions of the Heun equation with degree-dependent exponent parameters. Using four basic solutions $\mathfrak{t}$ ($\mathfrak{t}$ = a, b, c, and d) as TFs one can construct a quartet of radial potentials of Heun class quantized by polynomials. They represent four exactly solvable reductions of the potential of the type $(1, {}^{+}1/2, {}^{-}1/2)$ in Ishkhanyan's classification scheme for the 'Lemieux-Bose' potentials.

Similarly the reciprocal transformation $z \to 1/z$ of the constructed HRef CSLEs results in a quartet of Heun-Reference (HRef) CSLEs associated with the Liouville potentials on the line. Again they represent four reductions of the Lemieux-Bose potential of the type $(1, 1, {}^{-}1/2)$ with all the eigenfunctions expressible in terms of polynomial solutions of the Heun equation.

Both radial $\mathscr{J}\mathrm{Ref}$ potential and its LTP supplement have confluent counter-parts with a Coulomb and respectively with the infinite parabolic barrier at +∞. It will be shown in separate studies that the latter potentials are also double-step shape-invariant whereas the quartets of their single-step $D\mathfrak{T}s$ are quantized by polynomial solutions of the confluent Heun equation.

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The authors declare no conflict of interest.

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Appendix A

Radial ∮Ref potential in Ginocchio's representation

To relate our current notation for the radial ∮Ref potential to that in [3] it seems useful to slightly modify Wu's arguments [4, 5] in support of the assertion that the constant-mass limit of Ginocchio's radial 'position-dependent mass' (PDM) potential is nothing but another representation for the radial potential exactly solvable in terms of a superposition of two hypergeometric series [1, 2].

First let us set

$$\underline{a}=\lambda^{-2}(\lambda^{-2}-1)\ \ \underline{b}=\lambda^{-2},\ \ c_1\equiv\underline{a}+\underline{b}=\lambda^{-4} \qquad (A1)$$

so radial ∮Ref potential (11) takes form

$$_1\underline{V}_\lambda[\underline{z};\underline{\lambda}_o,\underline{\mu}_o]\equiv{}_1\underline{V}[\underline{z};\underline{\lambda}_o,\underline{\mu}_o;\lambda^{-2}(\lambda^{-2}-1),\lambda^{-2},0]$$

$$=(\underline{\lambda}_o^2-\tfrac{1}{4})\frac{\lambda^4(1-\underline{z})}{\underline{z}\,N[\underline{z};\lambda^2]}+{}_1\underline{V}_\lambda[\underline{z};\tfrac{1}{2},\underline{\mu}_o] \qquad (A2)$$

with

$$N[\underline{z};\lambda^2]:=\underline{z}+\lambda^2(1-\underline{z}) \qquad (A3)$$

and the radial ∮Ref potential with no centrifugal barrier [49]:

$$_1\underline{V}_\lambda[\underline{z};\tfrac{1}{2},\underline{\mu}_o]=\frac{\lambda^4(\tfrac{1}{4}-\underline{\mu}_o^2)(1-\underline{z})}{N[\underline{z};\lambda^2]}-\frac{\lambda^4(1-\lambda^2)(1-\underline{z})^2}{4N^2[\underline{z};\lambda^2]} \qquad (A4)$$

$$-\frac{\lambda^4(1-\lambda^2)(1-\underline{z})}{N^2[\underline{z};\lambda^2]}\left[1-\frac{5\lambda^2(1-\underline{z})}{4N[\underline{z};\lambda^2]}\right]$$

representing the non-singular remainder of sum (A2). In following Wu [4], we then introduce the auxiliary variable

$$y=\sqrt{\frac{\underline{z}}{N[\underline{z};\lambda^2]}}\quad\text{for}\ \ 0<\underline{z}<1, \qquad (A5)$$

so, as indicated by his Eqs. (3.4.11) and (3.4.12),

$$\underline{z}\equiv\underline{z}[y;\lambda^2]=\lambda^2y^2/Y(y^2;\lambda^2) \qquad (A6)$$

and

$$\frac{d\underline{z}}{dy}=\frac{2\lambda^2y}{Y^2(y^2;\lambda^2)} \qquad (A7)$$

respectively, where

$$Y(y^2;\lambda^2)\equiv1+(\lambda^2-1)y^2 \qquad (A8)$$

It is worth mentioning that the letter *z* in [1, 2, 4, 5] is used in exactly the same sense as $\underline{z}$ here and thereby the variable in question differs from Ginocchio's variable (3.2) in [3]. Multiplying the square of (A5) by its reverse (A6) one finds that polynomials (A3) and (A8) are related via the reciprocal formula

$$N[\underline{z};\lambda^2]=\frac{\lambda^2}{Y[y^2;\lambda^2]} \qquad (A9)$$

Substituting (A5) and

$$1-\underline{z}=(1-y^2)/Y(y^2;\lambda^2) \qquad (A10)$$

into the first derivative of $\underline{z}(r;\lambda)$ with respect to r,

$$\underline{z}'(r;\lambda^2)={}_1\underline{\rho}_\lambda^{-1/2}[\underline{z}]=2\lambda^2(1-\underline{z})\sqrt{\frac{\underline{z}}{N[\underline{z};\lambda^2]}} \qquad (A11)$$

and then making use of the chain rule coupled with (A7) shows that variable y(x) obeys the ODE

$$y'(r;\lambda)=(1-y^2)Y(y^2;\lambda^2)\quad\text{for}\ \ 0<y<+1. \qquad (A12)$$

Integrating the ODE

$$\frac{dr}{dy}=\frac{1}{\sqrt{(1-y^2)Y(y^2;\lambda)}} \qquad (A13)$$

under the boundary r=0 at y=0 we come to Eqs (2.8a) and (2.8b) in [49], with r varying from 0 to ∞ for 0 ≤ y < 1. Making use of (A3) and (A5) one can verify that the cited equations are equivalent to (15)-(17) in Section 2, with

$$r_1[\underline{z}]=\lambda^{-2}\mathit{arctanh}\,y[\underline{z}] \qquad (A14)$$

$$r_2[\underline{z}]=\begin{cases}\dfrac{\sqrt{\lambda^2-1}}{\lambda^2}\mathit{arctan}\left(\sqrt{\lambda^2-1}\,y[\underline{z}]\right) & \text{for}\ \lambda>1,\\[2ex] -\dfrac{\sqrt{1-\lambda^2}}{\lambda^2}\mathit{arctanh}\left(\sqrt{1-\lambda^2}\right) & \text{for}\ \lambda<1.\end{cases} \qquad (A15)$$

Taking advantage of (A9) we can alternatively represent (A10) as

$$\frac{1-\underline{z}}{N[\underline{z};\lambda^2]}=\lambda^{-2}(1-y^2) \qquad (A16)$$

Substituting (A9) and (A15) into (A4) thus gives

$$V_G[y;\tfrac{1}{2},\underline{\mu}_o;\lambda^2]\equiv{}_1\underline{V}_\lambda[\underline{z}[y];\tfrac{1}{2},\underline{\mu}_o]=$$

$$-\lambda^2(1-y^2)(\underline{\mu}_o^2-\tfrac{1}{4})+V_G[y;\tfrac{1}{2},\tfrac{1}{2};\lambda^2], \qquad (A17)$$

With

$$V_G[y;\tfrac{1}{2},\tfrac{1}{2};\lambda^2]=-\tfrac{1}{4}(1-\lambda^2)(1-y^2)\times$$

$$\left\{4Y[y^2;\lambda^2]-(1-y^2)\left[5Y[y^2;\lambda^2]+1\right]\right\} \qquad (A18)$$

Elementary algebraic modification then confirm that potential (A17) coincides with the symmetric Ginocchio potential on the line [49] if we choose

$$\underline{\mu}_o=\nu+\tfrac{1}{2} \qquad (A19)$$

Ultimately substituting (A6) and (A16) into the first summand in sum (A2) gives

$$V_G[y;\underline{\lambda}_o,\underline{\mu}_o;\lambda^2]=(\underline{\lambda}_o^2-\tfrac{1}{4})(y^{-2}-1)Y(y^2;\lambda)+V_G[y;\underline{\mu}_o;\lambda^2],$$

(A20)

in agreement with (7.86) in [5] with $\alpha_l=\underline{\lambda}_o$ and $\nu_l=\underline{\mu}_o-\tfrac{1}{2}$.

Setting *a* = 0 in (3.2b) in [3] brings us to the first (positive) root in pair (44) while constraint (A.1) turns (37) into Ginocchio's algebraic formula (3.2a) for bound energy levels.

After being expressed in terms of the variable $\underline{z}(r;\lambda^2)$ the eigenfunctions of the radial Schrodinger equation with potential (A2) take form

$$\underline{\psi}_{\mathbf{c}n}[\underline{z};\underline{\lambda}_o,\underline{\mu}_o;\lambda^2]=\lambda^{-2}B_n\underline{z}^{\frac{1}{2}\underline{\lambda}_o}(1-\underline{z})^{\frac{1}{2}(\underline{\lambda}_{1;\mathbf{c}n}-1)}\times \qquad (A21)$$

$$N^{\frac{1}{2}}[\underline{z};\lambda^2]\,{}_1\underline{\rho}_\lambda^{-\frac{1}{4}}[\underline{z}]P_n^{(\underline{\lambda}_{1;\mathbf{c}n},\underline{\lambda}_o)}(2z-1)$$

with the normalizing factor given by (21) in [2]:

$$\lambda^2 B_n = \left[\frac{\lambda^2 \lambda_{1;cn} + \lambda_o + 2n+1}{\lambda_{1;cn}(\lambda_o + \lambda_{1;cn} + 2n+1)} \times \frac{\Gamma(\lambda_o + n+1)\Gamma(\lambda_{1;cn} + n+1)}{n!\Gamma(\lambda_o + \lambda_{1;cn} + n+1)} \right]^{-1/2}, \quad \text{(A22)}$$

where we also took into account that

$$\frac{1}{\lambda_{1;\dagger n}(\lambda_o,\mu_o;\lambda^2)} - \frac{1-\lambda^2}{\lambda_o + \lambda_{1;\dagger n}(\lambda_o,\mu_o;\lambda^2) + 2n+1} = \quad \text{(A23)}$$

$$\frac{\lambda^2 \lambda_{1;\dagger n}(\lambda_o,\mu_o;\lambda^2) + \lambda_o + 2n+1}{\lambda_{1;\dagger n}(\lambda_o,\mu_o;\lambda^2)[\lambda_o + \lambda_{1;\dagger n}(\lambda_o,\mu_o;\lambda^2) + 2n+1]}$$

Keeping in mind that x in [3] stands for the variable $2\underset{\rightarrow}{z} - 1$ here the reader can directly verify that Ginocchio's normalizing factor (3.5b) in [3] is simply another form of (A22).

To match Ginocchio's [3] formula (4.1a) for the scattering function with (17) in [1]:

$$S(k) = \rho^{\lambda_1(k)} \frac{\Gamma(1-\lambda_1(k))}{\Gamma(1+\lambda_1(k))} \frac{\Gamma(\alpha(k))\Gamma(\beta(k))}{\Gamma(\lambda_o + 1 - \alpha(k))\Gamma(\lambda_o + 1 - \beta(k))} \quad \text{(A24)}$$

With

$$\lambda_1(k) = -i\sqrt{c_1}\,k \equiv -i\,k/\lambda^2 \quad \text{(A25)}$$

standing for $\overline{\beta}$ in [3], note that, according to its definition (16), the function $r_1[\underset{\rightarrow}{z}]$ satisfies the asymptotic relation:

$$\lim_{\underset{\rightarrow}{z}\to 1} \left[(1-\underset{\rightarrow}{z}) exp(2 r_1[\underset{\rightarrow}{z}]/\sqrt{c_1}) \right] = 4\lambda^{-2} \quad \text{(A26)}$$

and therefore

$$\lim_{\underset{\rightarrow}{z}\to 1} \left[(1-\underset{\rightarrow}{z}) e^{2\lambda^2 (r[\underset{\rightarrow}{z}] - r_0)} \right] = 4\lambda^{-2} \quad \text{(A27)}$$

where

$$r_0 := r_2[1] = \begin{cases} \dfrac{\sqrt{\lambda^2-1}}{\lambda^2} arctan(\sqrt{\lambda^2-1}) & \text{for } \lambda > 1, \\ -\dfrac{\sqrt{1-\lambda^2}}{\lambda^2} arctanh(\sqrt{1-\lambda^2}) & \text{for } \lambda < 1. \end{cases} \quad \text{(A28)}$$

(There is a misprint in Ginocchio's definition (2.3f) of the latter parameter for $\lambda > 1$,) Taking into account that, according to (A16),

$$\lim_{\underset{\rightarrow}{z}\to 1} \left[(1 - y[\underset{\rightarrow}{z}]) e^{2\lambda^2 (r[\underset{\rightarrow}{z}] - r_0)} \right] =$$

$$\tfrac{1}{2} \lim_{\underset{\rightarrow}{z}\to 1} \left[(1-\underset{\rightarrow}{z}) e^{2\lambda^2 (r[\underset{\rightarrow}{z}] - r_0)} \right] = 2\lambda^{-2} \quad \text{(A29)}$$

we thus come to (2.3e) in [3] as expected. Comparing the above expression with (6) in [3] shows that the parameter ρ appearing in S-matrix element (17) in [1] can be represented as

$$\rho = 4\lambda^{-2} e^{2\lambda^2 r_0} \quad \text{(A30)}$$

or alternatively

$$\rho = e^{2\lambda^2 r_1} \quad \text{(A31)}$$

making use of Ginocchio's parameter (4.1b),

$$r_1 = r_0 - \lambda^{-2} ln(\lambda/2) \quad \text{(A32)}$$

Substituting (A1) into (9) in [2], with

$$f \equiv \underset{\rightarrow}{\mu}_o^2 - 1, \quad \text{(A33)}$$

and representing Ginocchio's parameter (4.1d) for $l = a = 0$ as

$$\overline{\nu}_0 \equiv \underset{\rightarrow}{\mu}(k) := \sqrt{\underset{\rightarrow}{\mu}_o^2 + \lambda^{-4} k^2 (1-\lambda^2)} \quad \text{(A34)}$$

One finds

$$2\alpha(k) = \lambda_1(k) + \underset{\rightarrow}{\lambda}_o + 1 - \underset{\rightarrow}{\mu}(k)$$

$$2\beta(k) = \lambda_1(k) + \underset{\rightarrow}{\lambda}_o + 1 + \underset{\rightarrow}{\mu}(k) \quad \text{(A35)}$$

Keeping in mind that

$$\Gamma(1 + i\sqrt{c_1}k) / \Gamma(1 - i\sqrt{c_1}k) = -\Gamma(i\sqrt{c_1}k) / \Gamma(-i\sqrt{c_1}k) \quad \text{(A36)}$$

we can represent S-matrix element (A24) in Ginocchio's form (4.1a) which completes the proof.

Appendix B

'Generalized' Ginocchio potential

Let us now confirm that the so-called 'generalized' Ginocchio potential defined via (8) in [6], with λ, s, and γ standing for $\underset{\rightarrow}{\lambda}_o + \tfrac{1}{2}$, $\underset{\rightarrow}{\mu}_o - \tfrac{1}{2}$, and $\lambda > 1$ here, is nothing but the radial $\mathcal{J}$Ref potential expressed in terms of the new variable

$$u(r) \equiv atanh\sqrt{\underset{\rightarrow}{z}(r)} \quad \text{(B1)}$$

Indeed taking into account that

$$\frac{1}{N[\underset{\rightarrow}{z};\lambda^2]} = 1 + \frac{1-\lambda^2}{\lambda^2 + sinh^2 u} \quad \text{(B2)}$$

we can formally represent radial $\mathcal{J}$Ref potential (A2) as

$${}_1\underset{\rightarrow}{V}_\lambda[tanh^2 u; \underset{\rightarrow}{\lambda}_o, \underset{\rightarrow}{\mu}_o] = \frac{\lambda^4(\underset{\rightarrow}{\lambda}_o^2 - \tfrac{1}{4})}{sinh^2 u}$$

$$\left(1 + \frac{1-\lambda^2}{\lambda^2 + sinh^2 u}\right) + {}_1\underset{\rightarrow}{V}_\lambda[tanh^2 u; \tfrac{1}{2}, \underset{\rightarrow}{\mu}_o] \quad \text{(B3)}$$

Substituting (B2) and

$$\frac{1-\underset{\rightarrow}{z}}{N[\underset{\rightarrow}{z};\lambda^2]} = \frac{1}{\lambda^2 + sinh^2 u} \quad \text{(B4)}$$

into (A4) then gives

$${}_1\underset{\rightarrow}{V}_\lambda[tanh^2 u; \tfrac{1}{2}, \underset{\rightarrow}{\mu}_o] = -\frac{\lambda^4(\underset{\rightarrow}{\mu}_o^2 + \tfrac{3}{4} - \lambda^2)}{\lambda^2 + sinh^2 u} -$$

$$\frac{\lambda^4(1-\lambda^2)}{4(\lambda^2 + sinh^2 u)} \times \left[\frac{3(3\lambda^2 - 1)}{\lambda^2 + sinh^2 u} - \frac{5\lambda^2(1-\lambda^2)}{(\lambda^2 + sinh^2 u)^2} \right] \quad \text{(B5)}$$

in agreement with (8) in [6].

After being expressed in terms of the variable $\underset{\rightarrow}{z}(r)$ the Jost solutions

$$f_\pm(r;k) \equiv f_\pm[\underset{\rightarrow}{z}(r);k] \quad \text{(B6)}$$

defined via the conventional asymptotic formulas

$$\lim_{r\to\infty}\left[e^{\mp ikr} f_{\pm}(r;k)\right]=1 \quad (B7)$$

take form:

$$f_{\pm}[z;k]=\sqrt[4]{{}_1T_2[z]/c_1}\; z^{-\frac{1}{2}\lambda_o}\left(\frac{1-z}{\rho}\right)^{\pm\frac{1}{2}\lambda_1(k)}$$

$$F[\alpha_{\pm}(k),\beta_{\pm}(k);1\pm\lambda_o;1-z] \quad (B8)$$

where

$$\alpha_{-}(k)+\alpha_{+}(k)=\beta_{-}(k)+\beta_{+}(k)=\lambda_o+1 \quad (B9)$$

$$\alpha_{\pm}(k)+\beta_{\pm}(k)\pm\lambda_1(k)=\lambda_o+1 \quad (B10)$$

$$\beta_{-}(k)-\alpha_{-}(k)=\alpha_{+}(k)-\beta_{+}(k)$$
$$=\mu(k):=\sqrt{\mu_o^2(k)+ak^2}, \quad (B11)$$

and the parameter ρ can be expressed in terms of Ginocchio's parameter r_1 via (A29). The Jost function and its complex conjugated counter-part are defined via (12.140) in [24]:

$$\mathfrak{J}_{\pm}(k):=2\lambda_o \lim_{r\to 0}[r^{\lambda_o-\frac{1}{2}} f_{\pm}(k;r)]\equiv$$

$$2\lambda_o \lim_{z\to 0}\left((bz)^{\frac{1}{2}\lambda_o-\frac{1}{4}} f_{\pm}[k;z]\right) \quad (B12)$$

which gives

$$\mathfrak{F}_{\pm}(k)=\frac{2\lambda_o}{\sqrt[4]{c_1}}(b)^{\frac{1}{2}\lambda_o}\rho^{\pm i\frac{1}{2}\sqrt{c_1}k}$$
$$F\left[\alpha_{\pm}(k),\beta_{\pm}(k);1\pm\lambda_1(k);1\right] \quad (B13)$$

With

$$F\left[\alpha_{\pm}(k),\beta_{\pm}(k);1\pm\lambda_1(k);1\right]=$$
$$\frac{\Gamma\left(1\pm\lambda_1(k)\right)\Gamma\left(1\pm\lambda_1(k)-\alpha_{\pm}(k)-\beta_{\pm}(k)\right)}{\Gamma\left(1\pm\lambda_1(k)-\alpha_{\pm}(k)\right)\Gamma\left(1\pm\lambda_1(k)-\beta_{\pm}(k)\right)} \quad (B14)$$

Note that the presented expression for the Jost function differs by the factor $2\lambda_o$ from (14) in [1] because the latter factor $2\lambda_o\equiv 2s-1$ was incorrectly dropped in the definition of the Jost function.

The corresponding S-matrix element is obtained via (12.154) in [24]:

$$S(k)=\mathfrak{F}_{-}(k)/\mathfrak{F}_{+}(k) \quad (B15)$$

used by us in [1] to derive (A24). On other hand, as already pointed to by Lévai et al [6] their
expression (21) for this element (with k- dependent parameters (17) and (18) standing for $\mu(k)-\frac{1}{2}$ and $\lambda_1(k)$ in our notation) differs from (A24) by the ill-defined factor $(\square 1)^{\lambda\square\square}$ with the real exponent $\lambda\equiv\lambda_o+\frac{1}{2}$. In contrary to the explanation of this discrepancy in [6], the error came from the fact that the Jost function and its counter-part were defined via (12.142) and (12.142a) in [24]:

$$\mathfrak{J}_{\pm}(k):=(k/2)^{\lambda_o-\frac{1}{2}}e^{-\frac{1}{2}i\pi(\lambda_o-\frac{1}{2})}\frac{\sqrt{\pi}}{\Gamma(\lambda_o)}\mathfrak{F}_{\pm}(k) \quad (B16)$$

and as a result, according to (12.154) in [24]

$$S(k)=e^{i\pi(\lambda_o-\frac{1}{2})}\mathfrak{J}_{-}(k)/\mathfrak{J}_{+}(k) \quad (B17)$$

which brings us back to (A24), in agreement with Ginocchio's results [3].

Appendix C

Krein representation of CWs formed by solutions of generic CSLE

The representation of the DC$\mathfrak{T}$s of a principal solution the Schrödinger equation in terms of the ratio of two KDs was initially studied by Bagrov and Samsonov [68]. They took advantage of this representation to show that DCT can be represented as a linear differential operator acting on solutions of the Schrödinger equation under consideration. In particular making use of the chain relations for Krein determinants allowed the cited authors to prove the equivalence of the mentioned linear operators to integral transformations in the Gelfand-Levitan formalism (see [69] for details and the references therein).

The purpose of this appendix is to obtain the explicit relation between the CW and KD formed by seed solutions $\phi_k[z]$ of the generic CSLE at the energies ε_k, where we disregard the dependence of both solutions and energies on the parameters $\Lambda_o;T_K$. Namely we prove below that

$$\mathrm{Wr}\{\phi_1[z],\ldots,\phi_{2j+\ell}[z]\}=\boldsymbol{\rho}^{j(j+\ell-1)}[z]$$
$$\mathcal{K}\{\phi_{k=1,\ldots,2j+\ell}[z];\varepsilon_{k=1,\ldots,2j+\ell}\} \quad (C1)$$

where

$$\mathcal{K}\{\phi_{k=1,\ldots,2j}[z];\varepsilon_{k=1,\ldots,2j}\}:=$$
$$\begin{vmatrix} \phi_1[z] & \ldots & \phi_{2j}[z] \\ \dot{\phi}_1[z] & \ldots & \dot{\phi}_{2j}[z] \\ \varepsilon_1\phi_1[z] & \ldots & \varepsilon_{2j}\phi_{2j}[z] \\ & \ldots & \\ \varepsilon_1^{j-1}\phi_1[z] & \ldots & \varepsilon_{2j}^{j-1}\phi_{2j}[z] \\ \varepsilon_1^{j-1}\dot{\phi}_1[z] & \ldots & \varepsilon_{2j}^{j-1}\dot{\phi}_{2j}[z] \end{vmatrix} \quad (C2)$$

and

$$\mathcal{K}\{\phi_{k=1,\ldots,2j+1}[z];\varepsilon_{k=1,\ldots,2j+1}\}:=$$
$$(-1)^j\begin{vmatrix} \phi_1[z] & \ldots & \phi_{2j+1}[z] \\ \dot{\phi}_1[z] & \ldots & \dot{\phi}_{2j+1}[z] \\ \varepsilon_1\phi_1[z] & \ldots & \varepsilon_{2j+1}\phi_{2j+1}[z] \\ & \ldots & \\ \varepsilon_1^{j-1}\dot{\phi}_1[z] & \ldots & \varepsilon_{2j+1}^{j-1}\dot{\phi}_{2j+1}[z] \\ \varepsilon_1^{j}\phi_1[z] & \ldots & \varepsilon_{2j+1}^{j}\phi_{2j+1}[z] \end{vmatrix}. \quad (C3)$$

To confirm (C1) we formally represent the $(2j+\ell)$-th derivative of each seed solution $\phi_k[z]$ with respect to its argument z as a superposition of $\phi_k[z]$ and its first derivative

$$\frac{d^{2j'}\phi_k}{d^{2j'}z} = \left(-\varepsilon_k\,\rho[z]\right)^{j'}\phi_k[z] + \qquad \text{(C4)}$$

$$\mathfrak{F}^{(0)}_{j'-1}[z;\varepsilon_k]\phi_k[z] + \mathfrak{G}^{(0)}_{j'-1}[z;\varepsilon_k]\dot{\phi}_k[z]$$

And

$$\frac{d^{2j'+1}\phi_k}{d^{2j'+1}z} = \left(-\varepsilon_k\,\rho[z]\right)^{j'}\dot{\phi}_k[z] + \qquad \text{(C5)}$$

$$\mathfrak{F}^{(1)}_{j'}[z;\varepsilon_k]\phi_k[z] + \mathfrak{G}^{(1)}_{j'-1}[z;\varepsilon_k]\dot{\phi}_k[z],$$

where the *z*-dependent polynomials in ε are defined via the following mixed recurrence relations:

$$\mathfrak{F}^{(1)}_{j'}[z;\varepsilon] = j'\,ld\,\rho[z]\left(-\rho[z]\varepsilon\right)^{j'} +$$

$$\dot{\mathfrak{F}}^{(0)}_{j'-1}[z;\varepsilon] - \mathfrak{G}^{(0)}_{j'-1}[z;\varepsilon]\left(I^{o}[z] + \rho[z]\varepsilon\right),$$

$$\mathfrak{G}^{(1)}_{j'-1}[z;\varepsilon] = \dot{\mathfrak{G}}^{(0)}_{j'-1}[z;\varepsilon_k] \qquad \text{(C6)}$$

and

$$\mathfrak{F}^{(0)}_{j'-1}[z;\varepsilon] = -\left\{\mathfrak{G}^{(1)}_{j'-2}[z;\varepsilon] + \left(-\rho[z]\varepsilon\right)^{j'-1}\right\}$$

$$I^{o}[z] - \rho[z]\varepsilon\mathfrak{G}^{(1)}_{j'-2}[z;\varepsilon],$$

$$\mathfrak{G}^{(0)}_{j'-1}[z;\varepsilon] = (j'-1)\,ld\,\rho[z]\left(-\rho[z]\varepsilon\right)^{j'-1} \qquad \text{(C7)}$$

$$+\mathfrak{F}^{(1)}_{j'-1}[z;\varepsilon] + \dot{\mathfrak{G}}^{(1)}_{j'-2}[z;\varepsilon]$$

Starting from

$$\mathfrak{F}^{(1)}_{0}[z;\varepsilon] = \mathfrak{G}^{(1)}_{-1}[z;\varepsilon] \equiv 0 \qquad \text{(C8)}$$

and

$$\mathfrak{F}^{(0)}_{1}[z;\varepsilon] = -I^{o}[z] - \varepsilon\rho[z],\ \mathfrak{G}^{(0)}_{0}[\xi;\varepsilon] \equiv 0 \qquad \text{(C9)}$$

Substituting (C6) and (C7) into the CW, eliminating all the summands which contain the monomials $\varepsilon_k^{j''}\phi_k[z]$ or $\varepsilon_k^{j''}\dot{\phi}_k[z]$ from the two upper rows, and also taking into account that

$$2\sum_{j'=1}^{j-1} j' + \ell j = j(j-\ell-1) \qquad \text{(C10)}$$

we come to (C1) as asserted.